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INTERSECTION PROPERTIES OF BALLS AND
SUBSPACES IN BANACH SPACES

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INTRODUCTION. This paper is devoted to a study of the intersection between a linear subspace and a finite family of balls in a real or complex Banach space. In [4] Alfsen and Effros studied some subspaces of Banach spaces called L-ideals. A subspace J of a Banach space A is called an L-ideal if there exists a linear projection P , called an L-projection, in A such that $P(A) = J$ and

$$\|x\| = \|P(x)\| + \|x - P(x)\| \quad \text{all } x \in A$$

They said that a closed subspace J is an M-ideal if its annihilator J° is an L-ideal. We shall say that a linear subspace J of a Banach space A has the n intersection property (n. I.P.) if given n balls $\{B(a_i, r_i)\}_{i=1}^n$ in A such that $J \cap B(a_i, r_i + \epsilon) \neq \emptyset$ all i , all $\epsilon > 0$, and $\bigcap_{i=1}^n B(a_i, r_i + \epsilon) \neq \emptyset$ all $\epsilon > 0$, then $J \cap \bigcap_{i=1}^n B(a_i, r_i + \epsilon) \neq \emptyset$ all $\epsilon > 0$. The main result in [4] is the equivalence between the following statements:

- (i) J is an M-ideal
- (ii) J has the 3.I.P.
- (iii) J has the n.I.P. for all n .

A closed subspace J of A is called a Chebyshev subspace if for each $x \in A$ there exists a unique $y \in J$ such that

$$\|x - y\| = \inf\{\|x - z\| : z \in J\}$$

To each Chebyshev space J we can define a projection P by $P(x) = y$ if x and y are related as above. This projection P is usually non linear, and it is called the metric projection of A onto J . It is easy to see that each L-ideal is a Chebyshev subspace. We study some subspaces called semi L-ideals, which are more general than L-ideals, and we show [Theorem 3.6] that a closed

subspace J of A is a semi L -ideal if and only if J is a Chebyshev space and the metric projection P of A onto J behave like L -projections i.e.

$$\|x\| = \|P(x)\| + \|x - P(x)\| \quad \text{all } x \in A.$$

We call a closed subspace J a semi M -ideal if J° is a semi L -ideal. In Theorem 3.6. we show that J is a semi M -ideal if and only if it has the 2.I.P.

Our proofs are based on two separation lemmas of a linear subspace and a finite family of balls [Lemma 2.1. and Lemma 2.2.]. Thus since our proofs are algebraic we can dualize and show that J is a semi L -ideal if and only if J° is a semi M -ideal [Theorem 3.7]. These results easily extends to L -ideals and we give a new proof for that J is an M -ideal if and only if J has the 3.I.P. In fact, we show directly that a property formally weaker than the 3.I.P. for J implies that J is an M -ideal without first proving that 3.I.P. \Rightarrow n.I.P. all n [Theorem 3.13]. Dualized this gives that J is an L -ideal if and only if J° is an M -ideal [Theorem 3.14.]. This solves problem 1. and 2. of Alfsen and Effros [4]. In Theorem 3.3. we show that semi L -ideals in $L_1(\mu)$ spaces are L -ideals. In Corollary 4.2. we show that if A is the self-adjoint part of a C^* -algebra \mathcal{U} with unit and J is a semi M -ideal in A , then J is the self-adjoint part of a two sided ideal in \mathcal{U} . In the proof of Corollary 4.2. we use a result of Størmer [32] about Archimedean ideals. Cor. 4.2. ^{was} proved by Alfsen and Effros, Proposition 9.18. in [4] for M -ideals.

We show in section 1. that problem 3 of Alfsen and Effros [4] has a negative solution, i.e. the ϵ in the definition of the n.I.P. can not be taken to be 0 even if $n = 2$. In spite of this, we show that if J has the 3.I.P. and the Banach space A is an

almost $E(3)$ space then we can take ϵ in the definition of the n.I.P. for J to be 0 if we consider only two different balls [Proposition 1.2.]. (A Banach space is said to be an almost $E(n)$ space if for every family of n balls $\{B(a_i, r_i)\}_{i=1}^n$ in A with the property that $\bigcap_{i=1}^n B(\varphi(a_i), r_i) \neq \emptyset$ in \mathbb{R} or \mathbb{C} for all φ in the unit ball of A^* , we have

$$\bigcap_{i=1}^n B(a_i, r_i + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0.$$

In case A is real, this is the same as the n.2.I.P. in [25].) Proposition 1.2. is then used to give a new and simple proof for the following result of Hirsberg and Lazar [18]: If A is a Banach space with an extreme point e in the unit ball and A^* is isometric to a $L_1(\mu)$ space, then A is isometric to a subspace of the complex (real in case A is real) affine continuous functions on a compact convex set in such a way that e corresponds to 1. [Theorem 1.11.]. The main step in the proof of this result is to prove Corollary 1.9. which says that if A is an almost $E(3)$ space, J is an M -ideal in A and e is an extreme point in the unit ball of A , then the distance from e to J is 1.

In section 2 we prove two separation lemmas, Lemma 2.1. and Lemma 2.2. These results are used to give new characterizations of almost $E(n)$ spaces [Corollary 2.7.]. This corollary is then used to give a new proof [Theorem 2.12.] for the following result of Lindenstrauss [25] (real case) and Hustad [19] (complex case):

A^* is isometric to a $L_1(\mu)$ space if and only if A is an almost $E(n)$ space for all n . In Theorem 2.8. we show that the complex l_1 space is not an almost $E(3)$ space and we give a new proof for that the real l_1 space is an almost $E(3)$ space. The real case is first proved by Lindenstrauss [25], and the complex case was open.

NOTATION: Let A be a real or complex Banach space. Denote the real numbers by \mathbb{R} and the complex numbers by \mathbb{C} . \mathbb{K} will denote either \mathbb{R} or \mathbb{C} . $B(a, r)$ will denote the closed ball in A with center a and radius $r > 0$. The closed unit ball in A will be written A_1 and A^* will be the dual space of A . J will denote a linear subspace of A and $\varphi: A \rightarrow A/J$ will be the canonical map. If $a \in A$ and $S \subseteq A$, $S \neq \emptyset$, then we write the distance from a to S

$$d(a, S) = \inf\{\|a-s\| : s \in S\}$$

Thus we get the quotient norm on A/J $\|\varphi(a)\| = d(a, J)$.

If $S \subseteq A$, then $\text{co}(S)$ is the convex hull of S , and $\overline{\text{co}}(S)$ is the closed convex hull of S . If $S \subseteq A$ is convex, $\partial_e S$ will denote the set of extreme points in S .

Let $r = (r_j)_{j=1}^n \in \mathbb{R}^n$ be such that $r_j > 0$, $j = 1, n$. The n -product A^n will be considered with the following two norms:

$$(A^n, \|\cdot\|_{1,r}) = \{(a_1, a_n) \in A^n : \|(a_1, a_n)\| = \sum_{j=1}^n r_j \|a_j\|\}$$

and

$$(A^n, \|\cdot\|_{\infty,r}) = \{(a_1, a_n) \in A^n : \|(a_1, a_n)\| = \max\{r_j^{-1} \|a_j\| : j = 1, n\}\}$$

We will also consider the following subspace of $(A^n, \|\cdot\|_{1,r})$

$$H^n(A, J) = \{(a_1, a_n) \in A^n : \sum_{j=1}^n a_j \in J\}$$

The spaces $(\mathbb{K}^n, \|\cdot\|_{1,r})$ and $H^n(\mathbb{K}, (0))$ were closely studied in [19]. We will also write $H^n(A, (0)) = H^n(A)$.

We call A a Lindenstrauss space if A^* is isometric to a $L_1(\mu)$ space for some measure μ .

1. The Hirsberg - Lazar theorem.

This section is devoted to a simple proof of a representation theorem of Hirsberg and Lazar [18] for complex Lindenstrauss spaces whose unit balls have at least one extreme point. First we need some definitions.

DEFINITION. A finite family $\{B(a_j, r_j)\}_{j=1}^n$ of balls in A is said to have the weak intersection property if for any $\varphi \in A_1^*$

$$\bigcap_{j=1}^n B(\varphi(a_j), r_j) \neq \emptyset$$

in \mathbb{C} (in \mathbb{R} if A is a real Banach space).

PROPOSITION 1.1. Let $\{B(a_j, r_j)\}_{j=1}^n$ be a finite family of balls in A . Let \mathbb{K} be the scalar field of A . The following statements are equivalent:

(i) $\{B(a_j, r_j)\}_{j=1}^n$ has the weak intersection property.

(ii) $\left| \sum_{j=1}^n z_j \varphi(a_j) \right| \leq \sum_{j=1}^n |z_j| r_j$ for all $\varphi \in A_1^*$
and all $z = (z_1, z_n) \in H^n(\mathbb{K})$.

Proof: See [19] Corollary 1.3. and Corollary 1.4.

In [19] Hustad defined the notion of an almost $E(n)$ space. He gave a characterization of almost $E(n)$ spaces in terms of intersection properties of ball [Prop.1.13.]. Since we will be mainly concerned with intersection properties of balls we will take his characterization as our definition.

DEFINITION. We shall say that A is an almost $E(n)$ space if for any family $\{B(a_j, r_j)\}_{j=1}^n$ of n balls in A with the weak intersection property we have

$$(*) \quad \bigcap_{j=1}^n B(a_j, r_j + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0.$$

If we can take $\epsilon = 0$ in $(*)$ we shall say that A is an $E(n)$ space.

REMARK. In case A is a real Banach space, the weak intersection property of $\{B(a_j, r_j)\}_{j=1}^n$ is equivalent to $\|a_i - a_j\| \leq r_i + r_j$ for all $i, j = 1, \dots, n$. [See [19] Cor.1.10.]. Thus in case A is real, we get that A is an $E(n)$ space if and only if A has the n.2.I.P. (See [19] Cor.1.11. and [25])

REMARK. In [19] and [25] it is shown that A is a Lindenstrauss space if and only if A is an $E(n)$ space for all n . Moreover, every Banach space is a $E(2)$ space.

DEFINITION. We shall say that a subspace J of A has the n.I.P. (n -intersection property) if for every family $\{B(a_j, r_j)\}_{j=1}^n$ of n balls in A with the properties

$$(i) \quad J \cap B(a_j, r_j + \epsilon) \neq \emptyset \quad j = 1, \dots, n \quad \text{and all } \epsilon > 0$$

and

$$(ii) \quad \bigcap_{j=1}^n B(a_j, r_j + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0,$$

we have

$$(**) \quad J \cap \bigcap_{j=1}^n B(a_j, r_j + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0.$$

We shall say that J has the R.n.I.P. (restricted n.I.P.) if this holds for every family $\{B(a_j, r_j)\}_{j=1}^n$ where all $r_j = 1$.

If we can take $\epsilon = 0$ in (i), (ii) and (**), then we shall say that J has the strong n.I.P.

REMARK. It is clear that J has the strong n.I.P. $\Rightarrow J$ has the n.I.P. In [4] Alfsen and Effros gave an example of a subspace J such that J has the \exists .I.P. but not the strong \exists .I.P. Examining their example we see that J has the \exists .I.P. but not the strong 2.I.P. (In fact, if $v \in D_2 \cap D_3 \cap J$, then $0, k \leq k+p \leq v$. Since J^+ is hereditary, we get $k+p \in J$, so $p \in J$. But p is not compact, hence $D_2 \cap D_3 \cap J = \emptyset$.)

REMARK. The n.I.P. is equivalent to the property in (b) of Theorem 5.8. in [4].

PROPOSITION 1.2. Let $n \geq 1$ and let J be a closed subspace of A with the $(n+1)$.I.P. and assume A is an almost $E(n+1)$ space. Then J has the strong n.I.P.

Proof: Let $\{B(a_j, r_j)\}_{j=1}^n$ be a family of n balls in A such that

$$J \cap B(a_j, r_j + \epsilon) \neq \emptyset \quad j = 1, \dots, n, \quad \text{all } \epsilon > 0$$

and

$$\bigcap_{j=1}^n B(a_j, r_j + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0.$$

We will show that

$$J \cap \bigcap_{j=1}^n B(a_j, r_j) \neq \emptyset.$$

Let $\epsilon > 0$, let $\epsilon_m = \epsilon \cdot 2^{-m}$ and let $0 < \theta_m \leq \min\{\sqrt{r_j^2 + \epsilon_m^2} - r_j : j = 1, \dots, n\}$, $m = 1, 2, \dots$. Then it follows that if $a \in A$ with

$$\|a - a_j\| \leq \theta_m + r_j \quad j = 1, \dots, n$$

then $\{B(a_j, r_j)\}_{j=1}^n \cup \{B(a, \epsilon_m)\}$ has the weak intersection property. (See [19] Lemma 4.1.) A new proof will be given below.

The conclusion will now follow from an induction argument similar to that used in [19] Lemma 4.3. and [5].

Suppose we have found $(x_k)_{k=1}^p$ in A such that for $k = 1, \dots, p-1$

$$x_{k+1} \in J \cap B(x_k, \epsilon_k + \theta_{k+1}) \cap \bigcap_{j=1}^n B(a_j, r_j + \theta_{k+1}) .$$

Then $\{B(a_j, r_j)\}_{j=1}^n \cup \{B(x_p, \epsilon_p)\}$ has the weak intersection property. Now we use that A is an almost $E(n+1)$ space and then that J has the $(n+1)$.I.P. to find

$$x_{p+1} \in J \cap B(x_p, \epsilon_p + \theta_{p+1}) \cap \bigcap_{j=1}^n B(a_j, r_j + \theta_{p+1})$$

Now $(x_k)_{k=1}^\infty$ is a Cauchy sequence converging to some $x \in J \cap \bigcap_{j=1}^n B(a_j, r_j)$ and the proof is complete.

COROLLARY 1.3. Let J be a closed subspace of A with the 2.I.P. Let $a_1, a_2 \in A$ and $r_1, r_2 > 0$ such that $\|a_1 - a_2\| \leq r_1 + r_2$ and $d(a_i, J) \leq r_i$, $i = 1, 2$. If $\epsilon > 0$, then

$$J \cap B(a_1, r_1) \cap B(a_2, r_2 + \epsilon) \neq \emptyset$$

Proof: In the proof of Proposition 1.2. we can choose $0 < \theta_m \leq \epsilon_m$ and $x_1 \in J \cap B(a_1, r_1 + \epsilon) \cap B(a_2, r_2 + \epsilon)$. Then we continue as in the proof of Proposition 1.2. to find $(x_k)_{k=1}^\infty$ such that for $k \geq 1$

$$x_{k+1} \in J \cap B(a_1, r_1 + \theta_{k+1}) \cap B(x_k, \epsilon_k + \theta_{k+1})$$

Then $(x_k)_{k=1}^\infty$ converges to some

$$x \in J \cap B(a_1, r_1)$$

and $\|x-a_2\| \leq \|a_2-x_1\| + \sum_{k=1}^{\infty} \|x_{k+1}-x_k\| \leq 3\epsilon+r_2$, and the proof is complete.

COROLLARY 1.4. Let J be a closed subspace of A with the 2.I.P. Then

$$\varphi(A_1) = (A/J)_1.$$

LEMMA 1.5. Let J be a closed subspace of A with the 2.I.P. and let $\epsilon > 0$. If $x \in J$ with $\|x\| = 1$ and $a \in A_1$, then there exists $z \in J$ such that

$$\|x+a-z\| \leq 1+\epsilon \quad \text{and} \quad \|x-a+z\| \leq 1+\epsilon$$

Proof: We have $a \in B(a+x,1) \cap B(a-x,1)$, $x \in J \cap B(a+x,1)$ and $-x \in J \cap B(a-x,1)$. Let $z \in J \cap B(a+x,1+\epsilon) \cap B(a-x,1+\epsilon)$. This z fulfills the requirements. The proof is complete.

COROLLARY 1.6. Let J be a closed subspace of A with the strong 2.I.P. If $x \in J$ with $\|x\| = 1$ and $a \in A_1$, then there exists $z \in J$ such that

$$\|x+a-z\| \leq 1 \quad \text{and} \quad \|x-a+z\| \leq 1.$$

COROLLARY 1.7. Let J be a proper closed subspace of A with the strong 2.I.P. If $x \in J$, then $x \notin \partial_e A_1$.

Proof: Let $x \in J$ with $\|x\| = 1$ and let $a \in A_1 \setminus J$. Let z be as in Corollary 1.6. Then

$$x = \frac{1}{2}(x+a-z) + \frac{1}{2}(x-a+z)$$

so $x \notin \partial_e A_1$, and the proof is complete.

LEMMA 1.8. Let J be a closed subspace of A with the strong 2.I.P. If $F \subseteq A_1$ is a face such that $F \cap J = \emptyset$ and if $a \in F$, then

$$d(a, J) = 1.$$

Proof: Let $r = d(a, J) > 0$. Suppose $r < 1$. Then $(1-r)a \in B(0, 1-r) \cap B(a, r)$, $0 \in J \cap B(0, 1-r)$ and $J \cap B(a, r) \neq \emptyset$ by Corollary 1.3. Let

$$x \in J \cap B(0, 1-r) \cap B(a, r).$$

Then

$$1 = \|a\| \leq \|a-x\| + \|x\| \leq r + (1-r) = 1,$$

so $\|x\| = r$.

But then

$$a = r\left(\frac{x}{\|x\|}\right) + (1-r)\left(\frac{a-x}{\|a-x\|}\right) \in F$$

so

$$x(\|x\|)^{-1} \in F \cap J.$$

This contradiction shows that $r = 1$, and the proof is complete.

COROLLARY 1.9. Let J be a closed subspace of A with the strong 2.I.P. If $e \in \partial_e A_1$, then $d(e, J) = 1$.

Proof: Follows from Corollary 1.7. and Lemma 1.8.

THEOREM 1.10. Let A be an almost $E(3)$ space and suppose $e \in \partial_e A_1$. Denote $S = \{f \in A^* : \|f\| = 1 = f(e)\}$ and define $\Phi : A \rightarrow C(S)$ by $\Phi(x)(f) = f(x)$. If for all $p \in \partial_e A_1^*$, $J_p = \{x \in A : p(x) = 0\}$ has the 3.I.P. then Φ is an isometry into $C(S)$

such that $\Phi(e) = 1$.

Proof: We have that S is a w^* -closed face of A_1^* , and that $\|\Phi(x)\| \leq \|x\|$ for all $x \in A$. It is clear that $\Phi(e) = 1$. Let $x \in A$. We only have to prove that $\|\Phi(x)\| = \|x\|$. Let $p \in \partial_e A_1^*$ be such that $\|x\| = p(x)$. From Proposition 1.2. and Corollary 1.9. we get that $d(e, J_p) = 1$. Since the dual of A/J_p is isometric to $J_p^0 = \text{span}(p)$ we get that $|p(e)| = 1$. Hence for some $z \in \mathbb{C}$ with $|z| = 1$ we have $zp \in S$, so

$$\|x\| = |zp(x)| \leq \|\Phi(x)\|$$

and the proof is complete.

THEOREM 1.11. (Lindenstrauss, Hirsberg, Lazar.) Let A be a Lindenstrauss space and suppose the unit ball contains an extreme point e . Let S and Φ be as in Theorem 1.10. Then $\Phi(e) = 1$ and Φ is an isometry of A onto the w^* -continuous complex affine functions on S .

REMARK. The real case is proved by Lindenstrauss in [25]. The complex case is first proved by Hirsberg and Lazar in [18]. The onto argument can be found in [28] Theorem 18.

The theorem follows from Theorem 1.10, Theorem 2.12 (ii) and (iv) and Theorem 3.6.

We will now give a simple proof of Lemma 4.1. in [19].

LEMMA 1.12. (Hustad) Let A be a complex Banach space and let $\epsilon > 0$. Let $\{B(a_i, r_i)\}_{i=1}^n$ be n balls in A with the weak intersection property. If $a \in A$ satisfies

$$\|a - a_i\| \leq \sqrt{r_i^2 + \epsilon^2} \quad i = 1, \dots, n$$

then $\{B(a_i, r_i)\}_{i=1}^n \cup \{B(a, \epsilon)\}$ has the weak intersection property.

Proof: Let $a \in A$ with

$$\|a - a_i\| \leq \sqrt{r_i^2 + \epsilon^2} \quad i = 1, \dots, n.$$

By Corollary 3.7. in [19] it is enough to show that if $1 \leq i, j \leq n$, then $\{B(a_i, r_i), B(a_j, r_j), B(a, \epsilon)\}$ has the weak intersection property. So let $1 \leq i, j \leq n$ and let $f \in A_1^*$. Then

$$\begin{aligned} |f(a_i) - f(a_j)| &\leq r_i + r_j \\ |f(a_k) - f(a)| &\leq \sqrt{r_k^2 + \epsilon^2} \quad k = 1, \dots, n. \end{aligned}$$

If $B(f(a_i), r_i) \subseteq B(f(a_j), r_j)$ or $B(f(a_j), r_j) \subseteq B(f(a_i), r_i)$, then clearly $B(f(a_i), r_i) \cap B(f(a_j), r_j) \cap B(f(a), \epsilon) \neq \emptyset$ since

$\sqrt{r_k^2 + \epsilon^2} \leq r_k + \epsilon \quad k = 1, \dots, n$, so we may suppose $B(f(a_i), r_i)$ and $B(f(a_j), r_j)$ intersect in two different points E and F . Let S_j be that part of the plane containing $f(a_j)$ and determined by the lines from $f(a_i)$ through E and F . Let S_i be that part of the plane containing $f(a_i)$ and determined by the lines from $f(a_j)$ through E and F . If $f(a) \in S_i \cup S_j$, then

$$B(f(a_i), r_i) \cap B(f(a_j), r_j) \cap B(f(a), \epsilon) \neq \emptyset$$

since $\sqrt{r_k^2 + \epsilon^2} \leq r_k + \epsilon$, $k = i, j$.

The rest of the plane consists of two sectors T_1 and T_2 . Let T_1 be that sector determined by the lines through E and $f(a_i)$ and through E and $f(a_j)$. Suppose $f(a) \in T_1$. Then an inspection of the triangles $f(a_i)Ef(a)$ and $f(a_j)Ef(a)$, shows that in at least one of these triangles, the angle at E is between $\pi/2$ and π . Hence the distance from $f(a)$ to E is less than ϵ , so

$$E \in B(f(a_i), r_i) \cap B(f(a_j), r_j) \cap B(f(a), r).$$

The case $f(a) \in T_2$ is treated similarly.

The proof is complete.

2. A characterization of almost $E(n)$ spaces by extreme points.

We will now generalize Cor. 1.3. in [19] to arbitrary Banach spaces.

Let $r = (r_j)_{j=1}^n \in \mathbb{R}^n$ with all $r_j > 0$. It is easy to see that the dual of $(A^n, \|\cdot\|_{\infty, r})$ is isometric to $(A^{*n}, \|\cdot\|_{1, r})$, and that the dual of $(A^n, \|\cdot\|_{1, r})$ is isometric to $(A^{*n}, \|\cdot\|_{\infty, r})$.

Now we can prove:

LEMMA 2.1. Let J be a linear subspace of A , let $r_1, \dots, r_n > 0$ and let $a_1, \dots, a_n \in A$. The following statements are equivalent:

- (i) $J \cap \bigcap_{j=1}^n B(a_j, r_j + \epsilon) \neq \emptyset$ all $\epsilon > 0$
- (ii) $\left| \sum_{j=1}^n f_j(a_j) \right| \leq \sum_{j=1}^n r_j \|f_j\|$ all $(f_1, \dots, f_n) \in H^n(A^*, J^0)$
- (iii) $\left| \sum_{j=1}^n f_j(a_j) \right| \leq 1$ for all extreme points (f_1, \dots, f_n) of the unit ball of $H^n(A^*, J^0)$.

Proof: Define $\Delta(J, n) = \{(a_1, \dots, a_n) \in A^n : a_1 = \dots = a_n \in J\}$. Then $\Delta(J, n)$ is a linear subspace of $(A^n, \|\cdot\|_{\infty, r})$ with polar

$$\Delta(J, n)^0 = H^n(A^*, J^0)$$

Hence $H^n(A^*, J^0)$ is w^* -closed and

(ii) \iff (iii) follows since $(f_1, \dots, f_n) \rightarrow \sum_{j=1}^n f_j(a_j)$ is a w^* -continuous linear functional on $(A^{*n}, \|\cdot\|_{1, r})$.

(i) \implies (ii). Let $\epsilon > 0$ and let $a \in J \cap \bigcap_{j=1}^n B(a_j, r_j + \epsilon)$. Then

for $(f_1, \dots, f_n) \in H^n(A^*, J^0)$ we have

$$\left| \sum_{j=1}^n f_j(a_j) \right| = \left| \sum_{j=1}^n f_j(a_j - a) \right| \leq \sum_{j=1}^n \|f_j\| (r_j + \epsilon)$$

Since $\epsilon > 0$ is arbitrary, we get

$$\left| \sum_{j=1}^n f_j(a_j) \right| \leq \sum_{j=1}^n r_j \|f_j\|.$$

(ii) \Rightarrow (i) Here we will use a separation argument similar to one used in [26] p.348. (See also [21] and [22].)

Suppose that for some $\epsilon > 0$ we have

$$J \cap \bigcap_{j=1}^n B(a_j, r_j + \epsilon) = \emptyset.$$

Then for $0 < \theta < \min\{\epsilon \cdot r_j^{-1} : j = 1, \dots, n\}$ we have

$$\Delta(J, n) \cap B((a_1, \dots, a_n), 1 + \theta) = \emptyset$$

Hence $B((a_1, \dots, a_n), 1)$ and $\Delta(J, n)$ can be strongly separated. Let $(f_1, \dots, f_n) \in A^{*n}$ be such that

$$(2.1) \quad \sup_{j=1}^n \operatorname{Re} \left(\sum_{j=1}^n f_j(b_j) \right) < \inf_{x \in J} \operatorname{Re} \left(\sum_{j=1}^n f_j(x) \right)$$

$$(b_1, \dots, b_n) \in B((a_1, \dots, a_n), 1)$$

$$\text{Now } \sum_{j=1}^n f_j(x) = \operatorname{Re} \left(\sum_{j=1}^n f_j(x) \right) - i \operatorname{Re} \left(\sum_{j=1}^n f_j(ix) \right),$$

so from (2.1) it follows that if $x \in J$, then

$$\sum_{j=1}^n f_j(x) = 0$$

Hence $(f_1, \dots, f_n) \in H^n(A^*, J^0)$

We also have

$$\begin{aligned} \sup_{j=1}^n \operatorname{Re} \left(\sum_{j=1}^n f_j(b_j) \right) &= \sup_{\|y_j\| \leq r_j} \operatorname{Re} \left[\sum_{j=1}^n f_j(a_j) + \sum_{j=1}^n f_j(y_j) \right] \\ (b_1, \dots, b_n) &\in B((a_1, \dots, a_n), 1) \\ &= \operatorname{Re} \left(\sum_{j=1}^n f_j(a_j) \right) + \sup_{\|y_j\| \leq r_j} \operatorname{Re} \left(\sum_{j=1}^n f_j(y_j) \right) = \operatorname{Re} \left(\sum_{j=1}^n f_j(a_j) \right) + \sum_{j=1}^n \|f_j\| r_j \end{aligned}$$

Hence (2.1) gives

$$\sum_{j=1}^n r_j \|f_j\| < -\operatorname{Re} \left(\sum_{j=1}^n f_j(a_j) \right) \leq \left| \sum_{j=1}^n f_j(a_j) \right|$$

This contradicts (ii). Hence if (ii) is true then

$$J \cap \bigcap_{j=1}^n B(a_j, r_j + \epsilon) \quad \text{all } \epsilon > 0,$$

and the proof is complete.

A dual argument gives:

LEMMA 2.2. Let J be a closed subspace of A , let $r_1, r_n > 0$ and let $f_1, \dots, f_n \in A^*$. The following statements are equivalent:

- (i) $J^0 \cap \bigcap_{j=1}^n B(f_j, r_j + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0$
- (ii) $J^0 \cap \bigcap_{j=1}^n B(f_j, r_j) \neq \emptyset$
- (iii) $\left| \sum_{j=1}^n f_j(a_j) \right| \leq \sum_{j=1}^n r_j \|a_j\| \quad \text{all } (a_1, \dots, a_n) \in H^n(A, J)$

Proof: (i) \iff (ii) by w^* -compactness of the balls and since J^0 is w^* -closed.

(ii) \implies (iii) is proved as (i) \implies (ii) in Lemma 2.1. was proved.

(iii) \implies (ii). $B((f_1, \dots, f_n), 1)$ is w^* -compact in $(A^{*n}, \|\cdot\|_{\infty, r})$. Define

$$\Delta(J^0, n) = \{(g_1, \dots, g_n) \in A^{*n} : g_1 = \dots = g_n \in J^0\}.$$

Then

$$\Delta(J^0, n) = H^n(A, J)^0$$

so $\Delta(J^0, n)$ is w^* -closed.

If

$$J^0 \cap \bigcap_{j=1}^n B(f_j, r_j) = \emptyset$$

then $\Delta(J^0, n)$ and $B((f_1, \dots, f_n), 1)$ can be strongly separated by a ω^* -continuous linear functional. Now we proceed as in (ii) \Rightarrow (i) in Lemma 2.1. The proof is complete.

COROLLARY 2.3. Let J be a closed subspace of A , let $r_1, \dots, r_n > 0$ and let $a_1, \dots, a_n \in A$. Then the following statements are equivalent:

- (i) $J \cap \bigcap_{j=1}^n B(a_j, r_j + \epsilon) \neq \emptyset$ all $\epsilon > 0$
- (ii) $J^{00} \cap \bigcap_{j=1}^n B(a_j, r_j) \neq \emptyset$ in A^{**} .

Proof: Combine Lemma 2.1. and Lemma 2.2.

REMARK. Corollary 2.3. is a generalization of Lemma 5.8. in [25].

An argument similar to that used to prove Lemma 2.1. gives:

LEMMA 2.4. Let A be a real Banach space and let C be a convex cone in A . Let $r_1, \dots, r_n > 0$ and $a_1, \dots, a_n \in A$. The following statements are equivalent:

- (i) $C \cap \bigcap_{j=1}^n B(a_j, r_j + \epsilon) \neq \emptyset$ all $\epsilon > 0$
- (ii) $-\sum_{j=1}^n f_j(a_j) \leq \sum_{j=1}^n r_j \|f_j\|$ for all $f_1, \dots, f_n \in A^*$

such that $\sum_{j=1}^n f_j \geq 0$ on C .

If A is a real Lindenstrauss space then we can use Lemma 2.1. and that A^* has the Riesz decomposition property to give a short proof for the fact that real Lindenstrauss spaces is almost $E(n)$ spaces for all n . We will prefer to give a longer proof which is of a more general nature.

Let n be a natural number ≥ 2 . $H^n(A^*)_1$ is the unit ball of the subspace $H^n(A^*)$ of $(A^{*n}, \|\cdot\|_{1,r})$ where $r = (r_j)_{j=1}^n \in \mathbb{R}^n$ with all $r_j > 0$. We saw in the proof of Lemma 2.1. that $H^n(A^*)$ is ω^* -closed, and Lemma 2.1. shows that the extreme points of $H^n(A^*)_1$ is of some interest.

DEFINITION. Let A be a Banach space over the scalar field \mathbb{K} and let $n \geq 2$ be a natural number. Denote by S_n the following subset of $H^n(A^*)_1$:

$$S_n = \{f \in H^n(A^*)_1 : f = (z_1 g, \dots, z_n g) \text{ where } g \in A^* \text{ and } (z_1, \dots, z_n) \in H^n(\mathbb{K})\}$$

LEMMA 2.5. S_n is a ω^* -closed subset of $H^n(A^*)_1$.

Proof: Suppose $\{(z_1^\alpha g_\alpha, \dots, z_n^\alpha g_\alpha)\}_{\alpha \in I}$ is a net in S_n converging ω^* to $(f_1, \dots, f_n) \in (A^{*n}, \|\cdot\|_{1,r})$. Without loss of generality we may assume $\|g_\alpha\| = 1$ for all $\alpha \in I$. Then

$$1 \geq \|(z_1^\alpha g_\alpha, \dots, z_n^\alpha g_\alpha)\| = \sum_{j=1}^n r_j \|z_j^\alpha g_\alpha\| = \sum_{j=1}^n r_j |z_j^\alpha|$$

Using the ω^* -compactness of A_1^* (i.e. $\omega^* = \sigma(A^*, A)$) and the compactness of \mathbb{K}_1 , we may assume (going to a subnet if necessary) that $g_\alpha \rightarrow g \in A_1^*(\omega^*)$ and $z_j^\alpha \rightarrow z_j$.

Then

$$\sum_{j=1}^n z_j = \lim_{\alpha} \sum_{j=1}^n z_j^\alpha = 0$$

and

$$\sum_{j=1}^n r_j \|z_j g\| \leq \sum_{j=1}^n r_j |z_j| = \lim_{j=1}^n \sum_{j=1}^n r_j |z_j^\alpha| \leq 1$$

so

$$(z_1 g, \dots, z_n g) \in S_n$$

If $a_1, \dots, a_n \in A$, then

$$z_j^\alpha g_\alpha(a_j) \rightarrow z_j g(a_j) \quad j = 1, \dots, n.$$

Hence

$$(z_1^\alpha g_\alpha, \dots, z_n^\alpha g_\alpha) \rightarrow (z_1 g, \dots, z_n g) \quad w^*,$$

and the proof is complete.

We can now prove:

THEOREM 2.6. Let A be a Banach space, let $n \geq 2$ be a natural number and let $r_1, \dots, r_n > 0$. The following statements are equivalent:

- (i) If $\{B(a_j, r_j)\}_{j=1}^n$ are balls in A with the weak intersection property, then $\bigcap_{j=1}^n B(a_j, r_j + \epsilon) \neq \emptyset$ all $\epsilon > 0$.
- (ii) $\overline{\text{co}(S_n)}^{w^*} = H^n(A^*)_1$
- (iii) $\partial_e H^n(A^*)_1 \subseteq S_n$

Proof: (iii) \Rightarrow (ii) is obvious.

(ii) \Rightarrow (iii) follows from Lemma 2.5. and Milman's theorem [11, p.104]

(ii) \Rightarrow (i) Let $\{B(a_j, r_j)\}_{j=1}^n$ be balls in A with the weak intersection property. Let $(z_1 g, \dots, z_n g) \in S_n$ and assume $\|g\| = 1$. From Proposition 1.1. we get

$$\left| \sum_{j=1}^n z_j g(a_j) \right| \leq \sum_{j=1}^n |z_j| r_j = \sum_{j=1}^n r_j \|z_j g\| = \|(z_1 g, \dots, z_n g)\| \leq 1 .$$

Since

$$(f_1, \dots, f_n) \rightarrow \sum_{j=1}^n f_j(a_j)$$

is a w^* -continuous linear functional on $(A^{*n}, \|\cdot\|_{1,r})$ we get from (ii)

$$\left| \sum_{j=1}^n f_j(a_j) \right| \leq 1 \quad \text{all } (f_1, \dots, f_n) \in H^n(A^*)_1 .$$

Now Lemma 2.1. gives

$$\bigcap_{j=1}^n B(a_j, r_j + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0 .$$

(i) \Rightarrow (ii) Suppose $\overline{\text{co}(S_n)}^{w^*} \neq H^n(A^*)_1$. Then there exists $(f_1, \dots, f_n) \in H^n(A^*)_1$ with $(f_1, \dots, f_n) \notin \overline{\text{co}(S_n)}^{w^*}$. By Hahn-Banach (f_1, \dots, f_n) and $\overline{\text{co}(S_n)}^{w^*}$ can be strongly separated by an element $(a_1, \dots, a_n) \in A^n$. So we may assume

$$\text{Re} \left(\sum_{j=1}^n f_j(a_j) \right) > 1 \geq \sup_{(z_1 g, \dots, z_n g) \in S_n} \text{Re} \left(\sum_{j=1}^n z_j g(a_j) \right)$$

Hence

$$\left| \sum_{j=1}^n f_j(a_j) \right| > 1 \geq \left| \sum_{j=1}^n z_j g(a_j) \right| \quad \text{all } (z_1 g, \dots, z_n g) \in S_n .$$

Using Proposition 1.1. and Lemma 2.1. we see that $\{B(a_j, r_j)\}_{j=1}^n$ has the weak intersection property, and that for some $\epsilon > 0$

$$\bigcap_{j=1}^n B(a_j, r_j + \epsilon) = \emptyset$$

and the proof is complete.

COROLLARY 2.7. Let A be a Banach space and let $n \geq 2$ be a natural number. The following statements are equivalent:

(i) A is an almost $E(n)$ space.

(ii) For all $r_1, \dots, r_n > 0$ we have $\partial_e H^n(A^*)_1 \subseteq S_n$.

(iii) For $r_1 = \dots = r_n = 1$ we have $\partial_e H^n(A^*)_1 \subseteq S_n$.

Proof: We note that S_n and $H^n(A^*)_1$ depends on r_1, \dots, r_n .

(i) \iff (ii) follows from Theorem 2.6., and (ii) \implies (iii) is trivial.

(iii) \implies (i) follows from Theorem 2.6. and Lemma 1.12. in [19].

The proof is complete.

We will now use Corollary 2.7. on some special Banach spaces. First we will remark that in the real case Theorem 2.8. is proved in [25] Theorem 4.6. while the complex case has been unknown.

THEOREM 2.8. Let A be $l_1(\mathbb{K})$ or $l_1^k(\mathbb{K})$ where k is a natural number ≥ 2 . If $\mathbb{K} = \mathbb{R}$ then A is an almost $E(3)$ space, and if $\mathbb{K} = \mathbb{C}$ then A is not an almost $E(3)$ space.

Proof: The complex case. Let $r_1 = r_2 = r_3 = 1$. We will show that (iii) in Corollary 2.7. is not fulfilled. We have that A^* is \mathbb{C}^k with l_∞ -norm. Consider the following elements in $A^* = (\mathbb{C}^k, \|\cdot\|_\infty)$

$$x = (3^{-1}, 3^{-1}, \dots, 3^{-1}, \dots)$$

$$y = (3^{-1} e^{i\frac{2\pi}{3}}, 3^{-1} e^{-i\frac{2\pi}{3}}, \dots, 3^{-1} e^{-i\frac{2\pi}{3}}, \dots)$$

$$z = (3^{-1} e^{-i\frac{2\pi}{3}}, 3^{-1} e^{i\frac{2\pi}{3}}, \dots, 3^{-1} e^{i\frac{2\pi}{3}}, \dots)$$

Then $(x, y, z) \in H^3(A^*)$ and

$$\|(x, y, z)\| = \|x\| + \|y\| + \|z\| = 1$$

If $(x, y, z) \in S_3$, then for some $u \in A^*$ with $\|u\| = 1$ and some complex numbers a, b, c with $a + b + c = 0$ we have

$$(x, y, z) = (au, bu, cu)$$

$x = au$ gives that all coordinates of u are equal to $(3a)^{-1}$. Since $y = bu$ we get by considering the first and second coordinate in y that

$$3a \cdot 3^{-1} e^{i\frac{2\pi}{3}} = b = 3a \cdot 3^{-1} e^{-i\frac{2\pi}{3}}.$$

Hence, since $a \neq 0$,

$$e^{i\frac{2\pi}{3}} = e^{-i\frac{2\pi}{3}}.$$

This is a contradiction, so $(x, y, z) \notin S_3$.

Let (x_n, y_n, z_n) be the n -th coordinate of (x, y, z) . Then it follows from Theorem 3.6. in [19] that (x_n, y_n, z_n) is an extreme point in $H^3(\mathbb{C})_1$ for $n = 1, 2, \dots, k$. This now gives that $(x, y, z) \in \partial_e H^3(A^*)_1$, so (iii) and hence (i) in Corollary 2.7. with $n = 3$ is not fulfilled.

The real case. Let $r_1 = r_2 = r_3 = 1$. We will show that (iii) in Corollary 2.7. is fulfilled.

Let $(x, y, z) \in \partial_e H^3(A^*)_1$. We now have $A^* = (\mathbb{R}^k, \|\cdot\|_\infty)$. For $n = 1, \dots, k$ let (x_n, y_n, z_n) be the n -th coordinate of (x, y, z) . Without loss of generality we may assume $2^{-1} \geq \|x\| \geq \|y\| \geq \|z\| \geq 0$. We also have $1 = \|x\| + \|y\| + \|z\|$. Also we may assume $x_n \geq 0$ for all n . In fact, if u is such that $u_n = 1$ if $x_n \geq 0$ and $u_n = -1$ if $x_n < 0$, then $(ux, uy, uz) \in \partial_e H^3(A^*)_1$ and it is enough to show that $(ux, uy, uz) \in S_3$.

Claim: For each $n = 1, 2, \dots, k$ we have at least two equalities in

$$|x_n| \leq \|x\|, \quad |y_n| \leq \|y\|, \quad |z_n| \leq \|z\|.$$

Proof of claim: Suppose for example that $|x_p| < \|x\|$ and $|y_p| < \|y\|$. Choose $\epsilon > 0$ such that

$$|x_p \pm \epsilon| \leq \|x\|$$

$$|y_p \pm \epsilon| \leq \|y\| .$$

Then

$$z_p + (x_p \pm \epsilon) + (y_p \mp \epsilon) = 0 .$$

If we now change x and y at the p -th coordinate with $\pm \epsilon$, we see that $(x, y, z) \notin \partial_e H^3(A^*)_1$. This is a contradiction, so the claim is proved.

For each n we have that (x_n, y_n, z_n) is of one of the following three forms:

$$\text{I} \quad (x_n, y_n, z_n) = (\|y\| - \|z\|, -\|y\|, \|z\|)$$

$$\text{II} \quad (x_n, y_n, z_n) = (\|x\|, -\|x\| + \|z\|, -\|z\|)$$

$$\text{III} \quad (x_n, y_n, z_n) = (\|x\|, -\|y\|, -\|x\| + \|y\|)$$

Define elements $a, b, c, d, e \in A^*$ such that

$$a_n = \begin{cases} (2\|x\|)^{-1}(\|y\| - \|z\|) & \text{if I} \\ (2\|x\|)^{-1}(2\|x\| - \|y\| - \|z\|) & \text{if II or III} \end{cases}$$

$$b_n = \begin{cases} (2\|x\|)^{-1}(-\|y\|) & \text{if I or III} \\ (2\|x\|)^{-1}(-2\|x\| + 2\|z\| + \|y\|) & \text{if II} \end{cases}$$

$$c_n = \begin{cases} (2\|x\|)^{-1}\|z\| & \text{if I} \\ (2\|x\|)^{-1}(-\|z\|) & \text{if II} \\ (2\|x\|)^{-1}(-2\|x\| + 2\|y\| + \|z\|) & \text{if III} \end{cases}$$

$$d_n = \begin{cases} -2^{-1} & \text{if I} \\ 2^{-1} & \text{if II or III} \end{cases}$$

$$e_n = 2^{-1}$$

Then $(a, b, c) \in H^3(A^*)$ and $\|a\| = (2\|x\|)^{-1}(2\|x\| - \|y\| - \|z\|)$,

$\|b\| = (2\|x\|)^{-1}\|y\|$ and $\|c\| = (2\|x\|)^{-1}\|z\|$, so

$$\|(a,b,c)\| = \|a\| + \|b\| + \|c\| = 1 .$$

We now have

$$(x,y,z) = \|x\|(a,b,c) + \|y\|(e,-e,0) + \|z\|(d,0,-d)$$

Since $(x,y,z) \in \partial_e H^3(A^*)_1$, we have $\|z\| = 0$ and

$$(x,y,z) = (a,b,c) = (e,-e,0) \in S_3 .$$

The proof is complete.

DEFINITION. A linear projection e in a Banach space A is said to be an L-projection if

$$\|x\| = \|e(x)\| + \|x-e(x)\| \quad \text{all } x \in A .$$

A subspace J of A is said to be an L-ideal if J is the range of an L-projection.

L-projections have been studied in [9], [4], [10] and [17]. We will use L-projections together with Corollary 2.7. to give a new proof of the following implication: A is a Lindenstrauss space $\Rightarrow A$ is an almost $E(n)$ space for all n .

This implication is in fact an equivalence proved in the real case by Lindenstrauss [25] and in the complex case by Hustad [19]. Our proof will be similar in the real and complex case.

First we need two lemmas.

LEMMA 2.9. Let e be an L-projection in A^* and let $n \geq 2$. Let $(f_1, \dots, f_n) \in \partial_e H^n(A^*)_1$. Then $e(f_i) = f_i$ for all i or $e(f_i) = 0$ for all f_i .

Proof: Since $\sum_{j=1}^n f_j = 0$, we get $\sum_{j=1}^n e(f_j) = 0$. Hence $(e(f_1), \dots, e(f_n)), (f_1 - e(f_1), \dots, f_n - e(f_n)) \in H^n(A^*)$.

Thus

$$\begin{aligned} 1 &= \|(f_1, \dots, f_n)\| = \sum_{j=1}^n r_j \|f_j\| \\ &= \sum_{j=1}^n r_j \|e(f_j)\| + \sum_{j=1}^n r_j \|f_j - e(f_j)\| \\ &= \|(e(f_1), \dots, e(f_n))\| + \|(f_1 - e(f_1), \dots, f_n - e(f_n))\| \end{aligned}$$

Let $\alpha = \|(e(f_1), \dots, e(f_n))\|$. If $0 < \alpha < 1$ then we get a convex combination

$$\begin{aligned} (f_1, \dots, f_n) &= \alpha(\alpha^{-1}e(f_1), \dots, \alpha^{-1}e(f_n)) \\ &\quad + (1-\alpha)((1-\alpha)^{-1}(f_1 - e(f_1)), \dots, (1-\alpha)^{-1}(f_n - e(f_n))) \end{aligned}$$

This gives us a contradiction, so we must have $\alpha = 0$ or $\alpha = 1$, and the lemma follows.

From the proof of Lemma 2.9. it follows:

COROLLARY 2.10. Let e be an L -projection in a Banach space A and let $x \in \partial_e A_1$. Then $e(x) = x$ or $e(x) = 0$.

DEFINITION. Let A be a Banach space. For each L -projection e in A we define

$$N_e = \{x \in A_1 : e(x) = x \text{ or } e(x) = 0\}$$

and we define

$$N = \bigcap \{N_e : e \text{ is an } L\text{-projection in } A\}.$$

LEMMA 2.11. $[0, 1] \partial_e A_1 \subseteq N$.

Proof: Use Corollary 2.10.

EXAMPLE. Let $A = \mathbb{R}^3$ with the following norm. $\|(x,y,z)\| = \sqrt{x^2+y^2} + |z|$. Then $[0,1]\partial_e A_1 = N$.

This example shows that (ii) in Theorem 2.12. can not be weakend.

THEOREM 2.12. Let A be a Banach space with scalar field $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . The following statements are equivalent:

- (i) A is a Lindenstrauss space
- (ii) $[0,1]\partial_e A^*_1 = N$ and $\text{span}(x)$ is an L -ideal for all $x \in \partial_e A^*_1$.
- (iii) If $n \geq 2$, $r_1, \dots, r_n > 0$ and $(f_1, \dots, f_n) \in \partial_e H^n(A^*)_1$, then there exists $(z_1, \dots, z_n) \in \partial_e H^n(\mathbb{K})_1$ and $g \in \partial_e A^*_1$ such that $(f_1, \dots, f_n) = (z_1 g, \dots, z_n g)$
- (iv) A is an almost $E(n)$ space for all n .
- (v) A^{**} is an $E(n)$ space for all n .
- (vi) Any family of closed balls in A^{**} with the weak intersection property has a non-empty intersection.

Proof: (i) \Rightarrow (ii) From Lemma 2.11. it follows that

$$[0,1]\partial_e A^*_1 \subseteq N.$$

By definition A^* is isometric to a $L^1(X, \mu)$ space. If $B \subseteq X$ is a measurable subset, then we can define an L -projection e_B by

$$e_B(f) = f \cdot \chi_B$$

where χ_B is the characteristic function to B .

Also it is known [11, p.104] that the extreme points in the unit ball

of $L^1(X, \mu)$ are exactly the functions $z \mu(B)^{-1} \chi_B$ where $z \in \mathbb{K}$ with $|z| = 1$ and B is an atom. From this (ii) follows.

(ii) \Rightarrow (iii) Let $n \geq 2$, $r_1, \dots, r_n > 0$ and let $(f_1, \dots, f_n) \in \partial_e H^n(A^*)_1$. By Lemma 2.9. we have $e(f_i) = f_i$ or $e(f_i) = 0$ for all i and all L -projections in A^* . Hence $f_1, \dots, f_n \in [0, \infty) \partial_e A^*_1$. Let $g_i \in \partial_e A^*_1$ and $t_i \in \mathbb{K}$ be such that $f_i = t_i g_i$ for $i = 1, \dots, n$. Let e_i be the L -projection onto $\text{span}(f_i) = \text{span}(g_i)$. Then $e_i(g_i) = g_i$, so from Lemma 2.9. we get $e_i(g_j) = g_j$ for $j = 1, \dots, n$. Hence we may assume $g_1 = \dots = g_n = g \in \partial_e A^*_1$. Clearly $(t_1, \dots, t_n) \in H^n(\mathbb{K})_1$, and it is easy to see that if $(t_1, \dots, t_n) \notin \partial_e H^n(\mathbb{K})_1$, then $(f_1, \dots, f_n) \notin \partial_e H^n(A^*)_1$. Hence we must have $(t_1, \dots, t_n) \in \partial_e H^n(\mathbb{K})_1$ and the proof is complete.

(iii) \Rightarrow (iv) follows from Corollary 2.7.

(iv) \Rightarrow (v) Let $\{B(a_i, r_i)\}_{i=1}^n$ be n balls in A^{**} with the weak intersection property.

Suppose

$$\bigcap_{i=1}^n B(a_i, r_i) = \emptyset.$$

By Lemma 2.2. there exists $f_1, \dots, f_n \in A^*$ with $\sum_{i=1}^n f_i = 0$ such that

$$\left| \sum_{i=1}^n a_i(f_i) \right| > \sum_{i=1}^n r_i \|f_i\|$$

Let $\theta > 0$ such that

$$\left| \sum_{i=1}^n a_i(f_i) \right| > (1+\theta) \left(\sum_{i=1}^n r_i \|f_i\| \right)$$

Let $U = \text{span}(a_1, \dots, a_n)$ and $F = \text{span}(f_1, \dots, f_n)$. By the "principle of local reflexivity" (See [12] or [26].) there exists a linear operator $T : U \rightarrow A$ such that

$$T(a) = a \quad \text{if } a \in U \cap A$$

$$f(T(a)) = a(f) \quad \text{for } a \in U \text{ and } f \in F$$

$$(1-\theta)\|a\| \leq \|T(a)\| \leq (1+\theta)\|a\| \quad a \in U.$$

Now $\{B(T(a_i), (1+\theta)r_i)\}_{i=1}^n$ has the weak intersection property. In fact, if $\{z_i\}_{i=1}^n \subseteq \mathbb{K}$ with $\sum z_i = 0$, then

$$\left\| \sum_{i=1}^n z_i T(a_i) \right\| \leq \|T\| \left\| \sum_{i=1}^n z_i a_i \right\| \leq (1+\theta) \sum_{i=1}^n r_i |z_i|$$

and Corollary 1.4. in [19] shows that $\{B(T(a_i), (1+\theta)r_i)\}_{i=1}^n$ has the weak intersection property. By (iv) we have

$$\bigcap_{i=1}^n B(T(a_i), (1+\theta)r_i + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0.$$

But then Lemma 2.1. gives

$$(1+\theta) \left(\sum_{i=1}^n r_i \|f_i\| \right) < \left| \sum_{i=1}^n a_i(f_i) \right| = \left| \sum_{i=1}^n f_i(T(a_i)) \right| \leq \sum_{i=1}^n (1+\theta)r_i \|f_i\|$$

This contradiction shows that we must have $\bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset$ and (v) is proved.

(v) \Rightarrow (vi) follows from the w^{**} compactness of closed balls in A^{**} .

(vi) \Rightarrow (i) By the Theorem in [20] and Theorem 7.20. in [8] A^{**} is isometric to a $C(K)$ space where K is compact Hausdorff.

Hence by Proposition 1.18.1. and Corollary 1.13.3. in [30] (See [16] in the real case) A^* is isometric to a $L_1(\mu)$ space.

The proof is complete.

REMARK. The argument used in the proof of (iv) \Rightarrow (v) in Theorem 2.12. can be used to show that if A satisfies (d) in Corollary 1 to Theorem 5.4. in [25], then A^{**} also satisfies (d). Hence by Lemma 6.5. and Theorem 6.1. in [25] A satisfies (a) in Theorem 5.4. in [25].

LEMMA 2.13. Let J be a closed subspace of a real or complex Banach space A . Assume there exists a projection Q in A^* such

that $Q(A^*) = J^0$ and $\|I-Q\| \leq 1$. Then we have:

If $a_1, \dots, a_n \in J$, $r_1, \dots, r_n > 0$ and

$$\bigcap_{i=1}^n B(a_i, r_i + \epsilon) \neq \emptyset \text{ in } A \quad \text{all } \epsilon > 0,$$

then

$$J \cap \bigcap_{i=1}^n B(a_i, r_i + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0.$$

Proof: Let $f_1, \dots, f_n \in A^*$ such that $\sum_{i=1}^n f_i \in J^0$. Then we have $\sum_{i=1}^n (I-Q)(f_i) = 0$. Let $\epsilon > 0$ and let $a \in \bigcap_{i=1}^n B(a_i, r_i + \epsilon)$. Then

$$\begin{aligned} \left| \sum_{i=1}^n f_i(a_i) \right| &\leq \left| \sum_{i=1}^n Q(f_i)(a_i) \right| + \left| \sum_{i=1}^n (I-Q)(f_i)(a_i) \right| \\ &= \left| \sum_{i=1}^n (I-Q)(f_i)(a_i - a) \right| \\ &\leq \sum_{i=1}^n \|(I-Q)(f_i)\| \|a_i - a\| \\ &\leq \sum_{i=1}^n \|f_i\| (r_i + \epsilon) \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get

$$\left| \sum_{i=1}^n f_i(a_i) \right| \leq \sum_{i=1}^n \|f_i\| r_i.$$

By Lemma 2.1. we get

$$J \cap \bigcap_{i=1}^n B(a_i, r_i + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0$$

and the proof is complete.

THEOREM 2.14. Let J be a closed subspace of a real or complex Lindenstrauss space A . The following statements are equivalent:

(i) J is a Lindenstrauss space.

(ii) There exists a projection Q in A^* such that $Q(A^*) = J^0$ and $\|I-Q\| \leq 1$.

(iii) If $a_1, \dots, a_n \in J$ and $r_1, \dots, r_n > 0$ are such that

$$\bigcap_{i=1}^n B(a_i, r_i + \epsilon) \neq \emptyset \text{ in } A \quad \text{all } \epsilon > 0,$$

then

$$J \cap \bigcap_{i=1}^n B(a_i, r_i + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0.$$

Proof: (i) \Rightarrow (ii) follows from Lemma 17.3. in [22]. (That this lemma is valid in the complex case follows from results of Sakai [30]. See the preliminaries in [19], Theorem 7.20 in [8] and [13].)

(ii) \Rightarrow (iii) is just Lemma 2.13.

(iii) \Rightarrow (i) follows ^{from} Theorem 2.12. since (iii) clearly implies that J is an almost $E(n)$ space for all n .

The proof is complete.

REMARK. The proof of (i) \Rightarrow (ii) \Rightarrow (iii) is correct without the assumption that A is a Lindenstrauss space. We will show a result, Corollary 6.4., which gives a new proof of (i) \Rightarrow (ii).

Since every Banach space can be imbedded as a subspace of a $l_\infty(\Gamma)$ space for some set Γ , which is a Lindenstrauss space, we get:

COROLLARY 2.15. Let J be a Banach space. The following statements are equivalent:

(i) J is a Lindenstrauss space.

(ii) If A is any Banach space such that $J \subseteq A$, then there exists a projection P in A^* such that $P(A^*) = J^0$ and $\|I-P\| \leq 1$.

3. Semi L-ideals and semi M-ideals.

If $S \subseteq A_1$ we denote by $\text{face}(S)$ the smallest face of A_1 containing S . (See [2].)

If J is a closed subspace of A , the complementary cone J' is defined by

$$J' = \{x \in A : J \cap \text{face}(\|x\|^{-1}x) = \emptyset \text{ or } x = 0\}$$

From [4] Proposition 3.1. and [17] Theorem 1.2. we get the following proposition:

PROPOSITION 3.1. Let A be a Banach space with scalar field $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , and let J be a closed subspace. Then the following are equivalent:

- (i) J' is convex.
- (ii) J is an L-ideal.

Moreover, if J is an L-ideal and e is the unique L-projection onto J then $J' = (I-e)A$.

DEFINITION. Let A be a Banach space and let J be a closed subspace. We shall say that J is a semi L-ideal if

$$\|x+y\| = \|x\| + \|y\| \quad \text{all } x \in J, y \in J'.$$

REMARK. If J is an L-ideal, then J is a semi L-ideal. In fact, if e is the L-projection onto J and $x \in J, y \in J'$,

then

$$\|x+y\| = \|e(x+y)\| + \|(x+y) - e(x+y)\| = \|x\| + \|y\|$$

since $e(x) = x$ and $e(y) = 0$

We will give examples below which shows that the converse is false. But in a $L^1(\mu)$ space every semi L -ideal is an L -ideal.

LEMMA 3.2. Let J be a closed subspace of A . Then we have

$$\{y \in A : \|x+y\| = \|x\| + \|y\| \text{ all } x \in J\} \subseteq J'$$

Proof: From Theorem 2.9. in[4] we get that for every $y \in A$, there exists $y_1 \in J$ and $y_2 \in J'$ such that

$$y = y_1 + y_2, \quad \|y\| = \|y_1\| + \|y_2\|$$

If

$$\|y+x\| = \|x\| + \|y\| \quad \text{all } x \in J$$

then

$$\|y_2\| = \|y - y_1\| = \|y\| + \|y_1\|.$$

Hence $y = y_2 \in J'$, and the proof is complete.

THEOREM 3.3. Let A be a real or complex $L^1(\mu)$ space and let J be a semi L -ideal in A . Then J is an L -ideal.

Proof: From Proposition 3.1. it is enough to show that J' is convex. Let $g, h \in J'$ and let $f \in J$. Then $\|f+g\| = \|f\| + \|g\|$ and $\|f+h\| = \|f\| + \|h\|$. By Lemma 3.2. it is enough to show that

$$\|f+g+h\| = \|f\| + \|g+h\|.$$

We have $f, g, h \in A = L^1(X, \mu)$.

Hence

$$\begin{aligned} \int_X |f(x)+g(x)| d\mu(x) &= \|f+g\| \\ &= \|f\| + \|g\| = \int_X |f(x)| d\mu(x) + \int_X |g(x)| d\mu(x) \end{aligned}$$

Since $|f(x)+g(x)| \leq |f(x)| + |g(x)|$, we get

$$|f(x)+g(x)| = |f(x)| + |g(x)| \quad \text{a.e. } d\mu.$$

Let

$$B = \{x \in X : 0 < |f(x)| < \infty\}.$$

Then

$$|f(x)+g(x)| = |f(x)| + |g(x)| \quad \text{a.e. } d\mu \text{ on } B.$$

Hence

$$\frac{g(x)}{f(x)} \in [0, \infty) \quad \text{a.e. } d\mu \text{ on } B.$$

In the same manner we get

$$\frac{h(x)}{f(x)} \in [0, \infty) \quad \text{a.e. } d\mu \text{ on } B.$$

So

$$\frac{g(x)+h(x)}{f(x)} \in [0, \infty) \quad \text{a.e. } d\mu \text{ on } B,$$

and

$$|f(x)+g(x)+h(x)| = |f(x)| + |g(x)+h(x)| \quad \text{a.e. } d\mu \text{ on } B.$$

From this it follows that

$$\|f+g+h\| = \|f\| + \|g+h\|$$

and the proof is complete.

DEFINITION. Let A be a Banach space and let J be a closed subspace. Following [4] we shall say that J is an M-ideal if J° , the polar of J in A^* , is an L-ideal. Also we shall say

that J is a semi M-ideal if J^0 , the polar of J in A^* , is a semi L-ideal.

THEOREM 3.4. Let J be an M-ideal in a Banach space A . Then J has the n.I.P. for all n .

This theorem was first proved by Alfsen and Effros [4] Theorem 5.8. We will here give a short proof.

Proof: Let $a_1, \dots, a_n \in A$, $r_1, \dots, r_n > 0$ and $(f_1, \dots, f_n) \in H^n(A^*, J^0)$. Suppose

$$J \cap B(a_j, r_j + \epsilon) \neq \emptyset \quad j = 1, \dots, n \quad \text{and all } \epsilon > 0$$

and

$$\bigcap_{j=1}^n B(a_j, r_j + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0.$$

Let e be the L-projection onto J^0 and let $\epsilon > 0$. Let $x_j \in J \cap B(a_j, r_j + \epsilon)$ $j = 1, \dots, n$ and let $x \in \bigcap_{j=1}^n B(a_j, r_j + \epsilon)$

Now $e(f_i) \in J^0$, so $0 = \sum_{j=1}^n (f_j - e(f_j))$.

Hence

$$\begin{aligned} \left| \sum_{j=1}^n f_j(a_j) \right| &\leq \sum_{j=1}^n |e(f_j)(a_j)| + \left| \sum_{j=1}^n (f_j - e(f_j))(a_j) \right| \\ &= \sum_{j=1}^n |e(f_j)(a_j - x_j)| + \left| \sum_{j=1}^n (f_j - e(f_j))(a_j - x) \right| \\ &\leq \sum_{j=1}^n \|e(f_j)\| (r_j + \epsilon) + \sum_{j=1}^n \|f_j - e(f_j)\| (r_j + \epsilon) \\ &= \sum_{j=1}^n \|f_j\| (r_j + \epsilon) \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have

$$\left| \sum_{j=1}^n f_j(a_j) \right| \leq \sum_{j=1}^n r_j \|f_j\|.$$

From Lemma 2.1. it follows that

$$J \cap \bigcap_{j=1}^n B(a_j, r_j + \epsilon) \neq \emptyset \quad \text{all } \epsilon > 0$$

and the proof is complete.

In the same manner we get from Lemma 2.2.

THEOREM 3.5. Let J be an L -ideal in a Banach space A . Then J^0 has the n.I.P. for all n .

We will show that we have if and only if both in Theorem 3.4. and Theorem 3.5., but first we will characterize semi M -ideals.

THEOREM 3.6. Let J be a closed subspace of a real or complex Banach space A . The following statements are equivalent:

- (i) J is a semi M -ideal.
- (ii) J has the 2.I.P.
- (iii) J has the R.2.I.P.
- (iv) For all $\epsilon > 0$, all $x \in J$ with $\|x\| = 1$, all $y \in A_1$, there exists $z \in J$ with
$$\|x+y-z\| \leq 1+\epsilon, \quad \|x-y+z\| \leq 1+\epsilon$$
- (v) For all $f \in A^*$, there exists a unique $g \in J^0$ such that
$$\|f-g\| = d(f, J^0), \quad \text{and moreover}$$
$$\|f\| = \|g\| + \|f-g\|.$$

Dually we have:

THEOREM 3.7. Let J be a closed subspace of a real or complex Banach space A . The following statements are equivalent:

- (i) J^0 is a semi M -ideal
- (ii) J^0 has the 2.I.P.
- (iii) J^0 has the R.2.I.P.

(iv) For all $x \in J^0$ with $\|x\| = 1$, all $y \in A_1^*$, there exists $z \in J^0$ with

$$\|x+y-z\| = 1 = \|x-y+z\|$$

(v) For all $x \in A$, there exists a unique $y \in J$ such that $\|x-y\| = d(x, J)$, and moreover

$$\|x\| = \|y\| + \|x-y\|$$

(vi) J is a semi L -ideal.

Proof of Theorem 3.6. and Theorem 3.7.

(i) \Rightarrow (ii) in Theorem 3.6.

By definition J^0 is a semi L -ideal. Let $a_1, a_2 \in A$ and $r_1, r_2 > 0$ be such that $\|a_1 - a_2\| \leq r_1 + r_2$ and $d(a_i, J) \leq r_i$ $i = 1, 2$. Let $f_1, f_2 \in A^*$ be such that $f_1 + f_2 \in J^0$. Theorem 2.9. in [4] now gives that we can decompose

$$f_i = g_i + h_i, \quad g_i \in J^0, \quad h_i \in J^{0'}, \quad i = 1, 2.$$

This implies that

$$h_1 + h_2 = (f_1 + f_2) - (g_1 + g_2) \in J^0$$

Suppose now that there also exists $g_3 \in J^0$ and $h_3 \in J^{0'}$ such that $f_1 = g_3 + h_3$. Then

$$g_1 + h_1 = g_3 + h_3$$

and

$$\|h_1\| = \|h_3 + (g_3 - g_1)\| = \|h_3\| + \|g_3 - g_1\|$$

Similar we get

$$\|h_3\| = \|h_1\| + \|g_3 - g_1\|.$$

Hence $g_1 = g_3$ and $h_1 = h_3$

We have

$$\begin{aligned} f_1 &= g_1 + h_1 = g_1 + (f_1 + f_2) - (g_1 + g_2) - h_2 \\ &= (f_1 + f_2 - g_2) - h_2 \end{aligned}$$

and $f_1 + f_2 - g_2 \in J^0$ and $-h_2 \in J^0$.

Hence $h_1 = -h_2$.

Let $\epsilon > 0$ and let $b_i \in J \cap B(a_i, r_i + \epsilon)$ $i = 1, 2$.

Then

$$\begin{aligned} |f_1(a_1) + f_2(a_2)| &= |g_1(a_1) + g_2(a_2) + h_1(a_1) + h_2(a_2)| \\ &= |g_1(a_1) + g_2(a_2) + h_1(a_1 - a_2)| \\ &\leq |g_1(a_1 - b_1)| + |g_2(a_2 - b_2)| + |h_1(a_1 - a_2)| \\ &\leq \|g_1\|(r_1 + \epsilon) + \|g_2\|(r_2 + \epsilon) + \|h_1\|(r_1 + r_2) \\ &= \|f_1\|r_1 + \|f_2\|r_2 + \epsilon(\|g_1\| + \|g_2\|) \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have

$$|f_1(a_1) + f_2(a_2)| \leq r_1\|f_1\| + r_2\|f_2\|.$$

(ii) now follows from Lemma 2.1.

The proof of (i) \Rightarrow (ii) in Theorem 3.7. is similar and is omitted. Also (vi) \Rightarrow (ii) in Theorem 3.7. is proved this way. ^{only} We remark that we have to use Lemma 2.2. instead of Lemma 2.1. (ii) \Rightarrow (i) in Theorem 3.7. follows from (ii) \Rightarrow (i) in Theorem 3.6.

(ii) \Rightarrow (iii) in both theorems are trivial.

(iii) \Rightarrow (iv) in Theorem 3.6. is just Lemma 1.5. and

(iii) \Rightarrow (iv) in Theorem 3.7. is just Corollary 1.6. We only remark that J^0 is w^* -closed and the closed balls in A^* are w^* -compact so we don't need any ϵ in the conclusion.

The proof of (iv) \Rightarrow (v) \Rightarrow (i) in Theorem 3.6. is similar to the proof of (iv) \Rightarrow (v) \Rightarrow (vi) in Theorem 3.7. and is therefore omitted. Hence we will now concentrate on (iv) \Rightarrow (v) \Rightarrow (vi) in Theorem 3.7.

$$(v) \Rightarrow (vi)$$

Claim: $J' = \{z \in A : \|z\| = d(z, J)\}$.

Proof of claim. Suppose $x \in A \setminus J'$. By Theorem 2.9. in [4] there

exists $y \in J$, $z \in J'$ such that

$$x = y + z, \quad \|x\| = \|y\| + \|z\|.$$

Since $x \notin J'$ we have $y \neq 0$, so

$$\|x\| > \|z\| = \|x-y\| \geq d(x, J)$$

Hence

$$\{z \in A : \|z\| = d(z, J)\} \subseteq J'$$

If $x \in A$ and $\|x\| > d(x, J)$ there exists by (v) a unique $y \in J$ such that

$$\|x-y\| = d(x, J) < \|x\|, \quad \|x\| = \|y\| + \|x-y\|$$

Now if $x \in J'$, then by Lemma 2.7.(a) in [4] we get $0 \neq y \in J' \cap J$.

This is a contradiction, so $x \notin J'$ and the claim is proved.

We can now prove that J is a semi L -ideal. Let $x \in J$ and $y \in J'$ and denote $z = x + y$. By (v) there exists $u \in J$ such that

$$\|z-u\| = d(z, J), \quad \|z\| = \|u\| + \|z-u\|.$$

From the claim we get since $\|z-u\| = d(z-u, J)$,

$$\begin{aligned} \|z-u\| &= \|x+y-u\| \geq \|y\| = \|z-x\| \\ &= \|z-u+(u-x)\| \geq \|z-u\| \end{aligned}$$

Hence $\|y\| = \|z-u\| = \|z-x\|$, and by the uniqueness of $u \in J$ we get $u = x$. So

$$\|x+y\| = \|z\| = \|u\| + \|z-u\| = \|x\| + \|y\|,$$

and we have proved that J is a semi L -ideal.

It only remains to prove (iv) \implies (v). This will be proved in a series of 3 lemmas.

LEMMA 3.8. Suppose J^0 satisfy (iv) in Theorem 3.7. Let $x \in A$ and let $\epsilon > 0$. Suppose $y, z \in J$ and

$$\|x-y\| < d(x,J) + \epsilon , \quad \|x-z\| < d(x,J) + \epsilon$$

Then $\|y-z\| \leq 4\epsilon + 2\epsilon(\epsilon+d(x,J))$.

Proof: Since J^0 is isometric to the dual of A/J , we can find $f \in J^0$ with $\|f\| = 1$ such that

$$d(x,J) \geq f(x) = f(x-y) > \|x-y\| - \epsilon .$$

We can also find $g \in A^*$ with $\|g\| = 1$ such that

$$g(y-z) > \|y-z\| - \epsilon$$

From (iv) in Theorem 3.7. we get that there exists $h \in J^0$ such that

$$\|f+g-h\| \leq 1 + \epsilon , \quad \|f-g+h\| \leq 1 + \epsilon$$

(We use an ϵ here since it then will be easier to translate the proof to the situation in Theorem 3.6.)

It is obvious that

$$f(x-z) = f(x-y) + f(y-z) = f(x-y) > \|x-y\| - \epsilon$$

$$(f+g-h)(y-z) = g(y-z) > \|y-z\| - \epsilon$$

and

$$\|x-y\| \geq d(x,J) > \|x-z\| - \epsilon$$

We have

$$\operatorname{Re}(f+g-h)(x-z) \leq \|f+g-h\| \|x-z\| \leq (1+\epsilon)\|x-z\|$$

and

$$\begin{aligned} & \|x-y\| - \epsilon < f(x-y) \\ & = \frac{1}{2}(f+g-h)(x-y) + \frac{1}{2}(f-g+h)(x-y) \\ & = \frac{1}{2}\operatorname{Re}(f+g-h)(x-y) + \frac{1}{2}\operatorname{Re}(f-g+h)(x-y) \\ & \leq \frac{1}{2}\operatorname{Re}(f+g-h)(x-y) + \frac{1}{2}(1+\epsilon)\|x-y\| . \end{aligned}$$

This now gives

$$\operatorname{Re}(f+g-h)(x-y) > (1-\epsilon)\|x-y\| - 2\epsilon$$

and hence

$$\begin{aligned}
 & (1-\epsilon)\|x-y\| - 2\epsilon - (1+\epsilon)\|x-z\| \\
 & \leq \operatorname{Re}(f+g-h)(x-y) - \operatorname{Re}(f+g-h)(x-z) \\
 & = \operatorname{Re}(f+g-h)(z-y) \\
 & = (f+g-h)(z-y) = g(z-y) < \epsilon - \|y-z\|
 \end{aligned}$$

Hence

$$\begin{aligned}
 & \|y-z\| < 3\epsilon + (1+\epsilon)\|x-z\| - (1-\epsilon)\|x-y\| \\
 & < 3\epsilon + \|x-z\| + \epsilon\|x-z\| + (1-\epsilon)(\epsilon - \|x-z\|) \\
 & = 3\epsilon + 2\epsilon\|x-z\| + \epsilon(1-\epsilon) \\
 & < 4\epsilon + 2\epsilon(d(x,J)+\epsilon)
 \end{aligned}$$

and the lemma is proved.

LEMMA 3.9. Suppose J^0 satisfy (iv) in Theorem 3.7., and let $x \in A$. Then there exists a unique $y \in J$ such that $\|x-y\| = d(x,J)$.

Proof: Choose a sequence $(y_n)_{n=1}^{\infty}$ in J such that $\|x-y_n\| < d(x,J) + 2^{-n}$. From Lemma 3.8. it follows that $(y_n)_{n=1}^{\infty}$ is a Cauchy sequence. Hence it converges to an element $y \in J$. Clearly $\|x-y\| = d(x,J)$. The uniqueness follows from Lemma 3.8. and the proof is complete.

LEMMA 3.10. Suppose J^0 satisfy (iv) in Theorem 3.7. and let $x \in A$. Let y be the unique element in J such that $\|x-y\| = d(x,J)$. Then

$$\|x\| = \|y\| + \|x-y\|.$$

Proof: Put $z = x - y$. Then

$$\|z\| = \|x-y\| = d(x,J) = d(x-y,J) = d(z,J)$$

We can now choose sequences $(f_n)_{n=1}^{\infty} \subseteq J^0$ and $(g_n)_{n=1}^{\infty} \subseteq A^*$ such that $\|f_n\| = 1 = \|g_n\|$ for all n and

$$\|z\| \geq f_n(z) > \|z\| - n^{-1}$$

$$\|y\| \geq g_n(y) > \|y\| - n^{-1}$$

From (iv) in Theorem 3.7. there exists a sequence $(h_n)_{n=1}^{\infty} \subseteq J^0$ such that

$$\|f_n + g_n - h_n\| \leq 1 + n^{-1}, \quad \|f_n - g_n + h_n\| \leq 1 + n^{-1}$$

Since $y \in J$ and $f_n, h_n \in J^0$ we get

$$(f_n + g_n - h_n)(y) = g_n(y)$$

We have

$$|(f_n + g_n - h_n)(z)| \leq (1 + n^{-1})\|z\|$$

$$|(f_n - g_n + h_n)(z)| \leq (1 + n^{-1})\|z\|$$

so going to a subsequence if necessary, we may assume

$$(f_n + g_n - h_n)(z) \rightarrow a \in \mathbb{C} \text{ as } n \rightarrow \infty$$

$$(f_n - g_n + h_n)(z) \rightarrow b \in \mathbb{C} \text{ as } n \rightarrow \infty$$

and $|a| \leq \|z\|$, $|b| \leq \|z\|$.

Furthermore we have

$$\begin{aligned} \|z\| &= \lim f_n(z) \\ &= \lim \left(\frac{1}{2}(f_n + g_n - h_n)(z) + \frac{1}{2}(f_n - g_n + h_n)(z) \right) \\ &= \frac{1}{2}(a+b) = \frac{1}{2}|a+b| \leq \frac{1}{2}|a| + \frac{1}{2}|b| \leq \|z\|. \end{aligned}$$

Hence $2\|z\| = a+b$, so $a=b=\|z\|$.

But then

$$\begin{aligned} \|z\| + \|y\| &= |\lim(f_n + g_n - h_n)(z) + \lim(f_n + g_n - h_n)(y)| \\ &= \lim |(f_n + g_n - h_n)(x)| \\ &\leq \lim (1 + n^{-1})\|x\| = \|x\| \leq \|z\| + \|y\|, \end{aligned}$$

and the lemma is proved.

This also completes the proof of Theorem 3.6. and Theorem 3.7.

REMARK. Subspaces satisfying the conclusion in Lemma 3.9. is usually called Chebyshev. Thus we have proved that all semi L-ideals are Chebyshev.

COROLLARY 3.11. Let J be a closed subspace of A with the 2.I.P. Then every bounded linear functional of J has a unique norm-preserving extension to A .

Proof: This follows from (v) in Theorem 3.6.

COROLLARY 3.12. Let J be a closed subspace of A and assume J° has the 2.I.P. Then every ω^* continuous linear functional on J° has a unique norm-preserving extension to a ω^* continuous linear functional on A^* .

Proof: This follows from (v) in Theorem 3.7.

THEOREM 3.13. Let J be a closed subspace of a real or complex Banach space A . The following statements are equivalent:

- (i) J is an M-ideal.
- (ii) J has the n.I.P. for all n .
- (iii) J has the 3.I.P.
- (iv) J has the R.3.I.P.

THEOREM 3.14. Let J be a closed subspace of a real or complex Banach space A . The following statements are equivalent:

- (i) J° is an M-ideal.
- (ii) J° has the n.I.P. for all n .
- (iii) J° has the 3.I.P.
- (iv) J° has the R.3.I.P.
- (v) J is an L-ideal.

We will only prove Theorem 3.14. The proof of Theorem 3.13. is similar and will be omitted.

Proof: (i) \Rightarrow (ii) is just Theorem 3.4.

(ii) \Rightarrow (iii) \Rightarrow (iv) is trivial.

(iv) \Rightarrow (v). We will show that J' is convex. It then follows from Proposition 3.1. in [4] (and in case A is complex Theorem 1.2. in [17]) that J is an L -ideal.

Let $x_1, x_2 \in J'$. We have to show that $x_1 + x_2 \in J'$. From Theorem 3.7. we get that there exists $y \in J$ such that $\|x_1 + x_2 - y\| = d(x_1 + x_2, J)$. Put $x_3 = y - x_1 - x_2$. Then

$$\|x_3\| = d(x_1 + x_2, J), \quad \|y\| + \|x_3\| = \|x_1 + x_2\|.$$

In the proof of (v) \Rightarrow (vi) in Theorem 3.7. we showed

$$\{z \in A : \|z\| = d(z, J)\} = J'.$$

Since

$$\|x_3\| = d(x_1 + x_2, J) = d(y - x_1 - x_2, J) = d(x_3, J)$$

we get that $x_3 \in J'$.

Let $\epsilon > 0$. Since $\|x_i\| = d(x_i, J)$ $i = 1, 2, 3$, we can find $(f_i)_{i=1}^3 \subseteq J^0$ such that $\|f_i\| = 1$ $i = 1, 2, 3$ and

$$f_i(x_i) > \|x_i\| - \epsilon$$

We can also find $g \in A_1^*$ with

$$g(y) > \|y\| - \epsilon$$

Put $h_i = g + f_i$ $i = 1, 2, 3$.

Then

$$g \in \bigcap_{i=1}^3 B(h_i, 1)$$

and

$$f_i \in J^0 \cap B(h_i, 1) \quad i = 1, 2, 3.$$

From (iv) we get that there exists a

$$h \in J^0 \cap \bigcap_{i=1}^3 B(h_i, 1+\epsilon)$$

So

$$\begin{aligned} \|y\| - \epsilon + \sum_{i=1}^3 (\|x_i\| - \epsilon) &< g(y) + \sum_{i=1}^3 f_i(x_i) \\ &= g\left(\sum_{i=1}^3 x_i\right) + \sum_{i=1}^3 f_i(x_i) = \sum_{i=1}^3 (g+f_i)(x_i) \\ &= \sum_{i=1}^3 h_i(x_i) = \sum_{i=1}^3 h_i(x_i) - h(y) = \sum_{i=1}^3 (h_i - h)(x_i) \\ &\leq \sum_{i=1}^3 \|x_i\| (1+\epsilon) \end{aligned}$$

Hence

$$\|y\| < 4\epsilon + \epsilon \sum_{i=1}^3 \|x_i\| .$$

Since $\epsilon > 0$ is arbitrary, we get $y = 0$. So

$$x_1 + x_2 = -x_3 \in J'$$

(v) \Rightarrow (i) follows from Proposition 5.15. in [4].

The proof is complete.

REMARK. The equivalence of (i), (ii) and (iii) in Theorem 3.13. is proved by Alfsen-Effros in [4], Theorem 5.8. and Theorem 5.9.

(iv) is new.

(i) \Rightarrow (v) in Theorem 3.14. is stated as a problem in [4]. A partial proof of this is given in [10].

EXAMPLE. It is easy to prove that if B is a non-empty bounded subset of \mathbb{C} , then there exists a unique disc with minimum radius containing B .

(a) A semi M-ideal which is not an M-ideal.

Let $n \geq 3$, and let $x = (x_1, \dots, x_n) \in l_\infty^n(\mathbb{R})$.

There exists i and j such that

$$x_i \geq x_k \geq x_j \quad \text{all } k.$$

Put $\alpha = 2^{-1}(x_i + x_j)$. Then

$$\begin{aligned} d(x, \mathbb{R}(1, \dots, 1)) &= \|(x_1, \dots, x_n) - (\alpha, \dots, \alpha)\| \\ &= \max_k |x_k - \alpha| = 2^{-1} |x_i - x_j| \end{aligned}$$

and

$$\|(x_1, \dots, x_n) - (\alpha, \dots, \alpha)\| + \|(\alpha, \dots, \alpha)\| = 2^{-1} |x_i - x_j| + 2^{-1} |x_i + x_j| = \|x\|.$$

Since $J = \mathbb{R}(1, \dots, 1) = (H^n(\mathbb{R}))^0$ and J' is not convex,

$H^n(\mathbb{R})$ is a semi M-ideal and not an M-ideal.

A similar example is constructed in [4].

(b) A Chebyshev space that is not a semi L-ideal.

Let $n \geq 2$ and let $z = (z_1, \dots, z_n) \in l_\infty^n(\mathbb{C})$.

Let $\alpha \in \mathbb{C}$ be the unique number such that

$$r = d(z, \mathbb{C}(1, \dots, 1)) = \max_i |z_i - \alpha|$$

Then $B(z, r) \cap \mathbb{C}(1, \dots, 1) = (\alpha, \dots, \alpha)$, so $\mathbb{C}(1, \dots, 1)$ is a Chebyshev space. We will prove that $\mathbb{C}(1, \dots, 1)$ is not a semi L-ideal by proving that it does not satisfy (v) in Theorem 3.7. Let $z = (2, 2i) \in l_\infty^2(\mathbb{C})$ and put $\alpha = 1 + i$. Then $d(z, \mathbb{C}(1, 1)) = \|z - (\alpha, \alpha)\| = \sqrt{2}$ and $\|(\alpha, \alpha)\| = \sqrt{2}$. Hence

$$\|z - (\alpha, \alpha)\| + \|(\alpha, \alpha)\| = 2\sqrt{2} \neq 2 = \|z\|.$$

(c) Let X be a compact Hausdorff space and let $C_{\mathbb{R}}(X)$ be the Banach space of all real-valued continuous functions on X . Let J be the subspace spanned by 1. The same proof as in (a) shows that J is a semi L-ideal. Hence $J^0 = \{\mu \in C_{\mathbb{R}}(X)^* : \|\mu^+\| = \|\mu^-\|\}$ is a semi M-ideal.

THEOREM 3.15. Let X be a compact Hausdorff space containing at least three points. If e is an L -projection in $C_{\mathbb{R}}(X)$, then $e = 0$ or $e = I$.

Proof: Suppose $e: C_{\mathbb{R}}(X) \rightarrow C_{\mathbb{R}}(X)$ is an L -projection and $e \neq 0$. Since 1 is an extreme point in the unit ball of $C_{\mathbb{R}}(X)$, we get from Corollary 2.10. that $e(1) = 0$ or $e(1) = 1$. Without loss of generality, we may assume $e(1) = 1$.

We denote the dual of $C_{\mathbb{R}}(X)$ by $M(X)$ and the positive cone in $M(X)$ by $M^+(X)$. If $x \in X$ we denote the point measure at x by ϵ_x . Let e^* be the adjoint projection in $M(X)$. In [4] it is proved that (see Proposition 5.15.)

$$\|\mu\| = \max\{\|e^*(\mu)\|, \|\mu - e^*(\mu)\|\} \quad \mu \in M(X).$$

Let $f \in C_{\mathbb{R}}(X)$ and $f \geq 0$. Then

$$\|f\| \geq \| \|f\| - f \| = \| \|f\| - e(f) \| + \| f - e(f) \|$$

So

$$\|f\| \geq \| \|f\| - e(f) \|$$

Hence

$$e(f) \geq 0.$$

Thus for $\mu \in M^+(X)$ we have

$$e^*\mu(f) = \mu(e(f)) \geq 0$$

so $e^*\mu \geq 0$.

This gives

$$\|e^*\mu\| = e^*\mu(1) = \mu(e(1)) = \mu(1) = \|\mu\|.$$

Let $x \in X$. Then for some $\alpha \in [0, 1]$ and some $\mu_x \geq 0$ with $\|\mu_x\| = 1$ and $\mu_x(\{x\}) = 0$ we have

$$e^*\epsilon_x = \alpha \epsilon_x + (1-\alpha)\mu_x$$

Then

$$1 = \|\epsilon_X\| \geq \|\epsilon_X - e^*\epsilon_X\| = (1-\alpha)\|\epsilon_X\| + (1-\alpha)\|\mu_X\| = 2 - 2\alpha.$$

Hence $\alpha \geq 2^{-1}$.

Now

$$\begin{aligned} \alpha\epsilon_X + (1-\alpha)\mu_X &= e^*\epsilon_X = e^*(e^*\epsilon_X) \\ &= e^*(\alpha\epsilon_X + (1-\alpha)\mu_X) = \alpha^2\epsilon_X + \alpha(1-\alpha)\mu_X + (1-\alpha)e^*\mu_X \end{aligned}$$

so $\alpha = 1$ or

$$e^*\mu_X = \alpha\epsilon_X + (1-\alpha)\mu_X = e^*\epsilon_X$$

If $\alpha \neq 1$ then

$$1 = \|\mu_X\| \geq \|\mu_X - e^*\mu_X\| = \alpha\|\epsilon_X\| + \alpha\|\mu_X\| = 2\alpha$$

so $\alpha \leq 2^{-1}$.

Hence $\alpha = 1$ or $\alpha = 2^{-1}$.

Suppose $\alpha = 2^{-1}$. Since X contains at least three points, there exists $y \in X$ such that $x \neq y$ and $\epsilon_y \neq \mu_x$.

Now

$$e^*\epsilon_X = 2^{-1}\epsilon_X + 2^{-1}\mu_X.$$

In the same manner we get

$$e^*\epsilon_Y = 2^{-1}\epsilon_Y + 2^{-1}\mu_Y$$

for some $\mu_Y \geq 0$ with $\|\mu_Y\| = 1$. (We can have $\mu_Y = \epsilon_Y$.)

Now

$$2 = \|\epsilon_X - \epsilon_Y\| = \max\{\|e^*(\epsilon_X - \epsilon_Y)\|, \|(I - e^*)(\epsilon_X - \epsilon_Y)\|\}$$

and

$$\begin{aligned} 2\|e^*(\epsilon_X - \epsilon_Y)\| &= \|(\epsilon_X + \mu_X) - (\epsilon_Y + \mu_Y)\| \\ 2\|(I - e^*)(\epsilon_X - \epsilon_Y)\| &= \|(\epsilon_X + \mu_Y) - (\epsilon_Y + \mu_X)\| \end{aligned}$$

so

$$(\epsilon_X + \mu_X) \perp (\epsilon_Y + \mu_Y) \quad \text{or} \quad (\epsilon_X + \mu_Y) \perp (\epsilon_Y + \mu_X)$$

We have

$$\begin{aligned}\|e^*(\epsilon_x - \mu_x + \epsilon_y + \mu_y)\| &= \|\mu_y + \epsilon_y\| = 2 \\ \|(I - e^*)(\epsilon_x - \mu_x + \epsilon_y + \mu_y)\| &= \|\epsilon_x - \mu_x\| = 2.\end{aligned}$$

Hence

$$2 = \|\epsilon_x - \mu_x + \epsilon_y + \mu_y\|$$

so $\mu_x \not\perp \epsilon_y + \mu_y$.

But then we must have $(\epsilon_y + \mu_x) \perp (\epsilon_x + \mu_y)$, so

$$2 = \|\epsilon_x - \mu_x + \epsilon_y + \mu_y\| = \|\epsilon_x + \mu_y\| + \|\epsilon_y - \mu_x\| = 2 + \|\epsilon_y - \mu_x\|$$

Hence $\epsilon_y = \mu_x$.

This is a contradiction, so $\alpha = 1$. Hence $e^*\epsilon_x = \epsilon_x$ for all $x \in X$. Since e^* is w^* -continuous, we get $e^* = I$ and then $e = I$.

The proof is complete.

COROLLARY 3.16. Let A be a real Lindenstrauss space of dimension at least three. If e is an L -projection in A , then $e = 0$ or $e = I$.

Proof: By Proposition 5.15. in [4] the second adjoint e^{**} in A^{**} is an L -projection. In [16] it is proved that A^{**} is isometric to a $C_{\mathbb{R}}(X)$ space for some compact Hausdorff space X . By Theorem 3.15. $e^{**} = 0$ or $e^{**} = I$, so we must have $e = 0$ or $e = I$. The proof is complete.

We will now study how semi L -ideals and semi M -ideals behave with respect to the operations of taking sums and intersections. First we will show that semi L -ideals are hereditary.

DEFINITION. Let A be a Banach space and let C be a convex cone in A . We shall say that C is hereditary if $p \in A$, $q \in C$

and

$$\|q\| = \|p\| + \|q-p\|$$

implies $p \in C$.

PROPOSITION 3.17. Let A be a Banach space and let J be a closed subspace. If J is a semi L -ideal, then J is hereditary.

Proof: Let $q \in J$ and $p \in A$ and suppose

$$\|q\| = \|p\| + \|q-p\|$$

By Theorem 2.9. in [4] we may write

$$p = p_1 + p_2, \quad p_1 \in J, \quad p_2 \in J'.$$

Since J is a semi L -ideal and $q \in J$ we get

$$\|q\| = \|p_1\| + \|p_2\| + \|q-p\|$$

and

$$\|q\| + \|p_2\| = \|q-p_2\| = \|p_1 + (q-p)\| \leq \|p_1\| + \|q-p\|$$

Hence $p_2 = 0$ and $p = p_1 \in J$, and the proof is complete.

LEMMA 3.18. Let A be a Banach space and let J_1 and J_2 be two semi M -ideals in A . Then $J_1^0 + J_2^0$ is w^* closed in A^* .

Proof: Let $f \in J_1^0 + J_2^0$. Then $f = g + h$ where $g \in J_1^0$ and $h \in J_2^0$. By Theorem 2.9. in [4]

$$h = h_1 + h_2, \quad h_1 \in J_1^{0'}, \quad h_2 \in J_1^{0'}.$$

Since J_2^0 is a semi L -ideal, we get from Proposition 3.17. that $h_2 \in J_1^{0'} \cap J_2^0$. Hence we may write

$$f = (g + h_1) + h_2 \in J_1^0 + (J_2^0 \cap J_1^{0'})$$

Now let (f_α) be a net in $A_1^* \cap (J_1^0 + J_2^0)$.

Then we have

$$f_{\alpha} = g_{\alpha} + h_{\alpha}, \quad g_{\alpha} \in J_1^0, \quad h_{\alpha} \in J_2^0 \cap J_1^0$$

so

$$1 \geq \|f_{\alpha}\| = \|g_{\alpha}\| + \|h_{\alpha}\|$$

By w^* -compactness of A_1^* we may assume (going to subnets if necessary) that

$$g_{\alpha} \rightarrow g \in J_1^0, \quad h_{\alpha} \rightarrow h \in J_2^0 \quad (w^*)$$

Let $f = g + h \in J_1^0 + J_2^0$. Since the norm is w^* lower semi continuous we have

$$\|f\| \leq \|g\| + \|h\| \leq \liminf \|g_{\alpha}\| + \liminf \|h_{\alpha}\| \leq \liminf (\|g_{\alpha}\| + \|h_{\alpha}\|) \leq 1$$

So $f \in A_1^* \cap (J_1^0 + J_2^0)$. By the Krein-Smulian theorem [11] we have that $J_1^0 + J_2^0$ is w^* -closed, and the proof is complete.

COROLLARY 3.19. Let A be a Banach space and let J_1 and J_2 be two semi M -ideals in A . Then $J_1 + J_2$ is closed.

Proof: By a theorem of Reiter [29] we have $J_1 + J_2$ norm-closed if and only if $J_1^0 + J_2^0$ is w^* -closed, so the Corollary follows from Lemma 3.18.

PROPOSITION 3.20. Let A be a Banach space and let J_1 and J_2 be two semi L -ideals. Then $J_1 \cap J_2$ is a semi L -ideal and $(J_1 \cap J_2)' = (J_1 \cap J_2') + J_1'$.

Proof: Let $x \in A$. By Theorem 2.9. in [4] we can write

$$x = x_1 + x_2, \quad \|x\| = \|x_1\| + \|x_2\|, \quad x_1 \in J_1, \quad x_2 \in J_1'$$

Again Theorem 2.9. in [4] gives

$$x_1 = y_1 + y_2, \quad \|x_1\| = \|y_1\| + \|y_2\|, \quad y_1 \in J_2, \quad y_2 \in J_2'.$$

Since by Proposition 3.17. J_1 is hereditary we have

$$x = y_1 + y_2 + x_2 \in (J_1 \cap J_2) + (J_1 \cap J_2') + J_1'$$

and

$$\|x\| \leq \|y_1\| + \|y_2 + x_2\| \leq \|y_1\| + \|y_2\| + \|x_2\| = \|x_1\| + \|x_2\| = \|x\|$$

so

$$\|x\| = \|y_1\| + \|y_2 + x_2\|$$

Since $(J_1 \cap J_2)'$ is hereditary (See [4] Lemma 2.7.) we get that

$$(J_1 \cap J_2)' \subseteq (J_1 \cap J_2') + J_1'.$$

Suppose now that $x \in J_1 \cap J_2$ and $y = y_1 + y_2 \in (J_1 \cap J_2') + J_1'$. Then

$$\begin{aligned} \|x+y\| &= \|(x+y_1)+y_2\| = \|x+y_1\| + \|y_2\| \\ &= \|x\| + \|y_1\| + \|y_2\| = \|x\| + \|y_1+y_2\| = \|x\| + \|y\| \end{aligned}$$

since J_1 and J_2 are semi L-ideals. By Lemma 3.2.

$$(J_1 \cap J_2') + J_1' = (J_1 + J_2)'$$

and $J_1 \cap J_2$ is a semi L-ideal.

COROLLARY 3.21. Let J_1 and J_2 be two L-ideals in a Banach space A . Then $J_1 \cap J_2$ is an L-ideal.

Proof: Since J_1' and J_2' are convex, we have by Proposition 3.20. that $(J_1 \cap J_2)' = (J_1 \cap J_2') + J_1'$ is convex.

PROPOSITION 3.22. Let (J_α) be a family of semi M-ideals in a Banach space A . Then

$$\overline{\sum_{\alpha} J_{\alpha}}^{\text{norm}} \text{ is a semi M-ideal.}$$

Proof: Let $a_1, a_2 \in A$ and let $r_1, r_2 > 0$. Let $J = \overline{\sum_{\alpha} J_{\alpha}}^{\text{norm}}$ and suppose $\|a_1 - a_2\| \leq r_1 + r_2$ and $r_i \geq d(a_i, J)$ $i = 1, 2$. Let $\epsilon > 0$.

Then there exists $J_1, J_2 \in (J_\alpha)$ such that $r_i + 2^{-1}\epsilon \geq d(a_i, J_i)$. Since by Corollary 3.19. and Proposition 3.20. $J_1 + J_2$ is a semi-M-ideal, we have by Theorem 3.6.

$$(J_1 + J_2) \cap \bigcap_{i=1}^2 B(a_i, r_i + \epsilon) \neq \emptyset.$$

Hence

$$J \cap \bigcap_{i=1}^2 B(a_i, r_i + \epsilon) \neq \emptyset.$$

By Theorem 3.6. it follows that J is a semi M-ideal.

In the same manner we can give a new proof Proposition 5.11. a) in [4]:

COROLLARY 3.23. Let (J_α) be a family of M-ideals in a Banach space A . Then $\overline{\sum_\alpha J_\alpha}^{\text{norm}}$ is an M-ideal.

OBSERVATION. If J_1 and J_2 are semi M-ideals, then $J_1 \cap J_2$ need not be a semi M-ideal. Dually the sum of two semi L-ideals need not be a semi L-ideal.

Let $A = \mathbb{R}^3$ with l_1 -norm. Let

$$J_1 = \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 = 0\}$$

$$J_2 = \{(x_1, x_2, x_3) : x_1 = x_2 + x_3\}$$

In the example (a) just before Theorem 3.15. we showed that J_1 is a semi M-ideal. In the same manner we show that J_2 is a semi M-ideal. Now

$$J_1 \cap J_2 = \{(x_1, x_2, x_3) : x_1 = 0 = x_2 + x_3\}$$

Considering the balls $B((1, 0, 0), 1)$ and $B((1, 1, -1), 1)$ we see that $J_1 \cap J_2$ does not have the 2.I.P. By Theorem 3.6. $J_1 \cap J_2$ is not a semi M-ideal.

PROPOSITION 3.24. Let J_1 be an L-ideal and J_2 a semi L-ideal in a Banach space A . Then $(J_1+J_2)' = J_1' \cap J_2'$ and J_1+J_2 is a semi L-ideal.

Proof: As in the first part of Lemma 3.18. we show that

$$J_1 + J_2 = J_1 + (J_1' \cap J_2')$$

Let $(x_n)_{n=1}^{\infty}$ be a sequence in $J_1 + J_2$ converging to some element $x \in A$. Then (x_n) is a Cauchy sequence and we can write

$$x_n = y_n + z_n \in J_1 + (J_1' \cap J_2')$$

Since J_1 is an L-ideal we get

$$\|x_n - x_m\| = \|y_n - y_m\| + \|z_n - z_m\|$$

So both $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ are Cauchy sequences. Then there exists $y \in J_1$ and $z \in J_2$ such that

$$y_n \rightarrow y, \quad z_n \rightarrow z.$$

Hence $x = \lim x_n = y + z \in J_1 + J_2$, so $J_1 + J_2$ is closed.

Let $x \in A$. By using Theorem 2.9. in [4] we can split up x after J_1 and J_1' . Then we can split up the J_1' component after J_2 and J_2' . Since J_1' is hereditary we thus get

$$x = y + u + v \in J_1 + (J_1' \cap J_2) + (J_1' \cap J_2')$$

and

$$\|x\| = \|y+u\| + \|v\|$$

If $x \in (J_1+J_2)'$ then $y+u = 0$, so $x \in J_1' \cap J_2'$. If $x \notin (J_1+J_2)'$ then $y+u \neq 0$, so since $J_1' \cap J_2'$ is hereditary $x \notin J_1' \cap J_2'$.

Hence we have

$$(J_1+J_2)' = J_1' \cap J_2'.$$

Now let $x = y + u \in J_1 + (J_1' \cap J_2) = J_1 + J_2$, and let $z \in (J_1 + J_2)' = J_1' \cap J_2'$. Since J_1 and J_2 are semi L-ideals and J_1' is convex we get

$$\begin{aligned}\|x+z\| &= \|y+u+z\| = \|y\| + \|u+z\| \\ &= \|y\| + \|u\| + \|z\| = \|y+u\| + \|z\| = \|x\| + \|z\|.\end{aligned}$$

Hence $J_1 + J_2$ is a semi L-ideal and the proof is complete.

COROLLARY 3.25. Let J_1 be an M-ideal and J_2 a semi M-ideal in a Banach space A . Then $J_1 \cap J_2$ is a semi M-ideal.

Proof: $(J_1 \cap J_2)^0 = J_1^0 + J_2^0$. Now the result follows from proposition 3.24.

COROLLARY 3.26. Finite sums of L-ideals are L-ideals and finite intersections of M-ideals are M-ideals.

Proof: If J_1 and J_2 are L-ideals, then $J_1' \cap J_2'$ is convex, so $J_1 + J_2$ is an L-ideal by Proposition 3.24. Now the result follows by induction.

REMARK. Corollary 3.26. is proved in [4]. See also [9].

4. Semi M-ideals in order unit spaces.

In this section (A, e) will denote a real Banach space with a (positive) convex closed cone A^+ such that

- i) If $a \in A$, then there exists $t \geq 0$ such that $-te \leq a \leq te$
- ii) If $a \in A$ and $na \leq e$ all $n=1,2,\dots$, then $a \leq 0$
- iii) If $a \in A$ then $\|a\| = \inf\{\lambda > 0 : -\lambda e \leq a \leq \lambda e\}$

e is called order unit and $\| \cdot \|$ the order unit norm. Such a Banach space will be called an Archimedean order unit space. (See [1]p.67-69)

DEFINITION. A closed subspace of an Archimedean order unit space (A, e) is said to be a strongly Archimedean order ideal if it is satisfying:

- (i) If $\varphi: A \rightarrow A/J$ is the quotient map, if $\varphi(A^+)$ is the positive cone in A/J and if $n\varphi(a) \leq \varphi(e)$ all $n=1,2,\dots$ then $\varphi(a) \leq 0$.
- (ii) J is positively generated
- (iii) J^0 is positively generated.

THEOREM 4.1. Let J be a closed subspace in an Archimedean order unit space (A, e) . If J is a semi M -ideal, then J is a strongly Archimedean order ideal.

COROLLARY 4.2. Let A be the self-adjoint part of a C^* -algebra with unit. If J is a semi M -ideal in A , then J is an M -ideal.

Proof of corollary: The result follows from Theorem 4.1., Theorem 5.2. in [32] and Proposition 9.18. in [4]. (See also Proposition 7.1. in [3].)

Proof of Theorem 4.1.

- (a) J^0 is positively generated.

Let $x \in J^0$. By Proposition II.1.14. and Theorem II.1.15. in [1] we get that

$$x = y - z, \quad \|x\| = \|y\| + \|z\|$$

where $y, z \in A^{*+} = \{f \in A^* : f \text{ is positive on } A^+\}$.

Proposition 3.17. now gives that $y, z \in J^0$, so J^0 is positively generated.

(3) Let $\varphi : A \rightarrow A/J$ be the quotient map and let $\varphi(A^+)$ be the positive cone in A/J . Let $n\varphi(a) \leq \varphi(e)$ all $n = 1, 2, \dots$. Then $n^{-1}\varphi(e) - \varphi(a) \in \varphi(A^+)$ for all n and $n^{-1}\varphi(e) - \varphi(a) \rightarrow -\varphi(a)$ as $n \rightarrow \infty$. If we can show that $\varphi(A^+)$ is closed, then it follows that $\varphi(a) \leq 0$.

Claim 1: $\varphi(A^+) = \overline{\varphi(A^+)}$

Proof of claim 1: Let $x \in A$ and suppose $\varphi(x) \in \overline{\varphi(A^+)}$.

Define

$$K = \{f \in A^{*+} : \|f\| = f(e) = 1\}$$

and

$$F = J^0 \cap K$$

By Hahn-Banach we get $\varphi(x) \in \overline{\varphi(A^+)}$ if and only if $f(x) \geq 0$ for all $f \in J^0 \cap A^{*+}$.

Without loss of generality, we may assume $0 \leq x \leq 2e$ on F . Since J^0 is hereditary, F is a basis for $J^0 \cap A^{*+}$ and $-1 \leq f(x-e) \leq 1$ for all $f \in F$, we have $-1 \leq f(x-e) \leq 1$ for all f in the unit ball of J^0 . Since the dual of A/J is isometric to J^0 we get that

$$\|\varphi(x-e)\| \leq 1$$

i.e.

$$d(x-e, J) \leq 1 .$$

Corollary 1.3. now gives that there exists

$$y \in J \cap B(x-e, 1) .$$

Since A is equipped with order unit norm, we have

$$-e \leq x - e - y \leq e$$

Hence

$$0 \leq x - y .$$

So

$$\varphi(x) = \varphi(x-y) \in \varphi(A^+)$$

i.e. $\varphi(A^+)$ is closed.

Thus the claim and hence (i) in the definition of strongly Archimedean order ideal is proved.

(v) J is positively generated.

$$\text{Claim 2: } A_1^* \cap (J^0 + A^{*+}) \subseteq A^{*+} + (J^0 \cap A_1^*)$$

Proof of claim 2: Let $h \in A_1^* \cap (J^0 + A^{*+})$. Then

$$h = f + g, \quad f \in J^0, \quad g \in A^{*+}.$$

By Theorem 2.9. in [4] we may write

$$g = g_1 + g_2, \quad g_1 \in J^0, \quad g_2 \in J^{0'}$$

and

$$\|g\| = \|g_1\| + \|g_2\|.$$

Now since $g \in A^{*+}$ we have

$$\|g_1\| + \|g_2\| = \|g\| = g(e) = g_1(e) + g_2(e) \leq \|g_1\| + \|g_2\|.$$

Hence $\|g_2\| = g_2(e)$, so $g_2 \in A^{*+}$ (See [1] Corollary II.1.5.)

Since $f + g_1 \in J^0$ and J^0 is a semi L -ideal we have

$$1 \geq \|h\| = \|f + g_1\| + \|g_2\|$$

so $1 \geq \|f + g_1\|$ and

$$h \in A^{*+} + (J^0 \cap A_1^*)$$

and the claim is proved.

Let J^+ denote $J \cap A^+$. Taking polars in the formula in Claim 2 gives (See [6] Proposition 1.1.)

$$(4-1) \quad A^+ \cap (J + A_1) \subseteq \overline{(J \cap A^+) + A_1} = \overline{J^+ + A_1}$$

Let $x \in J$ with $\|x\| = 1$ and let $\epsilon > 0$.

From Corollary 1.3. and Theorem 3.6. we get that there exists some

$$y \in J \cap B(e-x, 1) \cap B(e+x, 1+\epsilon)$$

$y \in B(e-x, 1)$ gives

$$-e \leq e - x - y \leq e$$

so

$$0 \leq x + y \leq 2e.$$

$y \in B(e+x, 1+\epsilon)$ gives

$$-(1+\epsilon)e \leq e + x - y \leq (1+\epsilon)e$$

so

$$-(2+\epsilon)e \leq x - y \leq \epsilon e$$

Hence $\|x+y\| \leq 2$, $\|x-y\| \leq 2+\epsilon$ and

$$-y \leq x \leq y + \epsilon e$$

Thus

$$(4-2) \quad x = 2^{-1}(x+y) + 2^{-1}(x-y)$$

and $x+y \in J^+$ and $y-x \geq -\epsilon e$

Since $x, y \in J$ we get from (4-1)

$$y - x + \epsilon e \in A^+ \cap (J + A_\epsilon) \subseteq \overline{J^+ + A_\epsilon} \subseteq J^+ + A_{2\epsilon}$$

Choose $z \in J^+$ and $u \in A_{2\epsilon}$ such that

$$y - x + \epsilon e = z + u$$

Then $\|z\| \leq 2 + 3\epsilon$ and

$$y - x = z + (u - \epsilon e)$$

(4-2) now gives

$$\begin{aligned} (4-3) \quad x &= 2^{-1}(x+y) - 2^{-1}z - 2^{-1}(u - \epsilon e) \\ &= 2^{-1}(x+y) - (2+3\epsilon)^{-1}z - 2^{-1}(u - \epsilon e) + z((2+3\epsilon)^{-1} - 2^{-1}) \end{aligned}$$

We have

$$\begin{aligned} &\|2^{-1}(u - \epsilon e) + z(2^{-1} - (2+3\epsilon)^{-1})\| \\ &\leq 2^{-1} \cdot 3\epsilon + (2+3\epsilon)(2^{-1} - (2+3\epsilon)^{-1}) = 3\epsilon \end{aligned}$$

Hence from (4-3)

$$x \in (J^+ \cap A_1) - (J^+ \cap A_1) + A_{3\epsilon}$$

so

$$(4-4) \quad J \cap A_1 \subseteq \overline{(J^+ \cap A_1) - (J^+ \cap A_1)}$$

The Tykey-Klee- Ellis lemma (See [14] Lemma 7) and (4-4) gives

$$(4-5) \quad J \cap A_{1-\theta} \subseteq (J^+ \cap A_1) - (J^+ \cap A_1) \quad \text{all } 0 < \theta < 1 .$$

Hence J is positively generated, and the proof of Theorem 4.1. is complete.

REMARK. Let J be a semi M -ideal in (A, e) .

Let $x \in J$ with $\|x\| = 1$. Formula (4-5) is easily seen to imply that

$$J \cap B(e + \epsilon x, 1 + \epsilon) \cap B(e + \epsilon x, 1 + \epsilon) \neq \emptyset .$$

(In fact, if $x = y - z$, $y, z \in J^+ \cap A_{1+\epsilon}$ then the intersection contains $y + z$.)

On the other hand if x is the compact operator k constructed by Stefansson [31] (See also [4] Remark 5.10.) then an easy argument by contradiction gives

$$J \cap B(e + x, 1) \cap B(e - x, 1) = \emptyset .$$

(See also the remark before Proposition 1.2.)

LEMMA 4.3. Let J be a semi M -ideal in (A, e) . Then $J^0 + A^{*+}$ is ω^* -closed and

$$-(J \cap A^+)^0 = J^0 + A^{*+}$$

Proof: Let (h_α) be a net in $A_1^* \cap (J^0 + A^{*+})$ converging ω^* to some $h \in A_1^*$. As in the proof of claim 2 in the proof of Theorem 4.1. we may write

$$h_\alpha = g_\alpha + f_\alpha, \quad 1 \geq \|h_\alpha\| = \|g_\alpha\| + \|f_\alpha\|$$

where $f_\alpha \in J^0$ and $g_\alpha \in A^{*+} \cap J^{0'}$.

Since $J^0 \cap A_1^*$ and $A_1^* \cap A^{*+}$ both are ω^* -compact we may assume (going to a subnet if necessary) that

$$\begin{aligned} f_\alpha &\rightarrow f \in J^0 & \omega^* \\ g_\alpha &\rightarrow g \in A^{*+} & \omega^* . \end{aligned}$$

Using the ω^* lower semi-continuity of the norm, we get

$$\begin{aligned} \|h\| &= \|\lim h_\alpha\| = \|\lim(f_\alpha + g_\alpha)\| = \|f+g\| \leq \|f\| + \|g\| \\ &\leq \liminf \|f_\alpha\| + \liminf \|g_\alpha\| \leq \liminf (\|f_\alpha\| + \|g_\alpha\|) = \liminf \|h_\alpha\| \leq 1. \end{aligned}$$

Hence $h \in A_1^* \cap (J^0 + A^{*+})$.

By the Krein-Šmulian theorem $J^0 + A^{*+}$ is ω^* -closed.

The formula

$$-(J \cap A^+)^0 = J^0 + A^{*+}$$

now follows from Proposition 1.1. in [6] and the proof is complete.

COROLLARY 4.4. Let J be a semi M -ideal in (A, e) and let $a \in A^+$. Then

$$d(a, J) = d(a, J \cap A^+)$$

Proof: Suppose $r = d(a, J) > 0$. Let $f \in A^*$ and suppose $f \geq 0$ on $J \cap A^+$. Then $f \in -(J \cap A^+)^0 = J^0 + A^{*+}$. Thus we may write

$$f = g + h, \quad g \in J^0, \quad h \in A^{*+} \cap J^{0'}$$

and

$$\|f\| = \|g\| + \|h\|.$$

Now since $B(a, r) \cap J \neq \emptyset$ we get from the trivial part of Lemma 2.1. that

$$-f(a) = -g(a) - h(a) \leq -g(a) \leq r\|g\| \leq r\|f\|$$

Now Lemma 2.4. gives

$$B(a, r+\epsilon) \cap J \cap A^+ \neq \emptyset \quad \text{all } \epsilon > 0 ,$$

and the Corollary follows.

COROLLARY 4.5. Let J be a semi M -ideal in (A, e) . Then each bounded positive linear functional on J has a unique norm-preserving extension to a positive linear functional on A .

Proof: Let f be a bounded positive linear functional on J . Then $f \geq 0$ on $J \cap A^+$ and we may assume $\|f\| = 1$. By Corollary 3.11. f has a unique norm-preserving extension, denoted f to A . Now $f \in -(J \cap A^+)^0 = J^0 + A^{*+}$, so as before we write

$$f = g + h, \quad g \in J^0, \quad h \in A^{*+} \cap J^0,$$

and $1 = \|f\| = \|g\| + \|h\|$. Then we get

$$1 = \|f\| = \|f|_J\| = \|h|_J\| = d(h, J^0) = \|h\|$$

so $g = 0$ and $f = h \in A^{*+}$. The proof is complete.

REMARK. In [6] Theorem 2.2. Asimow has shown that there is a close connection between the statements in Corollary 4.4. and Corollary 4.5.

REMARK. In the example after Theorem 3.14. we showed that $H^3(\mathbb{R})$ is a semi M -ideal in $l_1^3(\mathbb{R})$ but not an M -ideal. Let $e = (1, 0, 0)$. If we give $l_1^3(\mathbb{R})$ the partial ordering defined in (3) in Theorem 4.7. in [25] then it follows that $(l_1^3(\mathbb{R}), e)$ is an Archimedean order unit space. Hence Corollary 4.2. is false in general.

5. Banach spaces with many M-ideals.

In section 2 we said that a linear projection e in A is an L-projection if

$$\|x\| = \|e(x)\| + \|x-e(x)\| \quad \text{all } x \in A.$$

A subspace of A is said to be an L-ideal if it is the range of an L-projection. If J is an L-ideal in A and e is the L-projection onto J , then $J' = (I-e)(A)$ and e is the unique L-projection with range J . See [4] Proposition 3.1. and [9] Lemma 2.1. Cunningham and Alfsen and Effros also shows that L-projections commute.

LEMMA 5.1. Let J and H be two L-ideals of A . Then $J \cap H$ is an L-ideal in H and

$$H = (J \cap H) \oplus (J' \cap H)$$

Proof: See Corollary 3.21., Proposition 3.14. in [4] or Lemma 2.3. in [9].

LEMMA 5.2. Let J be an L-ideal in A and let H be an L-ideal in J . Then H is an L-ideal in A .

Proof: Let $e: A \rightarrow J$ be the L-projection onto J and let $f: J \rightarrow H$ be the L-projection in J onto H . For $x \in A$ we have

$$\begin{aligned} \|x\| &= \|e(x)\| + \|x-e(x)\| = \|fe(x)\| + \|e(x)-fe(x)\| + \|x-e(x)\| \\ &= \|fe(x)\| + \|x-fe(x)\| \end{aligned}$$

Hence fe is an L-projection in A with range H , so H is an L-ideal in A and the proof is complete.

LEMMA 5.3. Suppose A contains infinitely many different L -ideals. Then there exists a sequence $(J_n)_{n=1}^{\infty}$ of L -ideals in A such that $J_n \not\subseteq J_{n+1}$ for all n and

$$A \supseteq J_1 \supseteq J_2 \supseteq \dots \supseteq J_n \supseteq J_{n+1} \supseteq \dots$$

Proof: Let J be an L -ideal in A such that $(0) \neq J \neq A$. Then it follows from Lemma 5.1. that J or J' contains infinitely many different L -ideals. Hence we may choose an L -ideal J_1 such that $(0) \neq J_1 \neq A$ and J_1 contains infinitely many L -ideals. An easy induction argument now gives the result.

REMARK. The argument used in Lemma 5.3. was suggested to me by Erik Alfsen.

c_0 will denote the Banach space of all real (complex if A is complex) sequences converging to 0 with supremum norm. l_1 is the dual of c_0 .

THEOREM 5.4. Suppose A contains infinitely many different L -ideals. Then there exists a linear operator $T: l_1 \rightarrow A$ such that

$$\|Tx\| = \|x\| \quad \text{all } x \in l_1.$$

Proof: By Lemma 5.3. there exists a sequence $(J_n)_{n=1}^{\infty}$ of L -ideals in A such that $J_n \not\subseteq J_{n+1}$ all n and

$$A \supseteq J_1 \supseteq J_2 \supseteq \dots \supseteq J_n \supseteq J_{n+1} \supseteq \dots$$

Since L -projections commute we have

$$J'_{n+1} \cap J_n \neq (0) \quad \text{all } n.$$

Hence we may choose a sequence $(x_n)_{n=1}^{\infty}$ in A such that

$$x_n \in J_n \cap J'_{n+1}, \quad \|x_n\| = 1 \quad \text{all } n.$$

Let e_n be the L -projection in A onto J_n . Let $c = (c_n) \in l_1$.

Then we have

$$\begin{aligned} \left\| \sum_{k=1}^n c_k x_k \right\| &= \|e_2(\sum_{k=1}^n c_k x_k)\| + \|(I-e_2)(\sum_{k=1}^n c_k x_k)\| \\ &= \left\| \sum_{k=2}^n c_k x_k \right\| + \|c_1 x_1\| \\ &= \|e_3(\sum_{k=2}^n c_k x_k)\| + \|(I-e_3)(\sum_{k=2}^n c_k x_k)\| + \|c_1 x_1\| \\ &= \left\| \sum_{k=3}^n c_k x_k \right\| + \|c_2 x_2\| + \|c_1 x_1\| \\ &= \dots = \sum_{k=1}^n \|c_k x_k\| = \sum_{k=1}^n |c_k| \end{aligned}$$

In particular

$$\left\| \sum_{k=n}^{n+m} c_k x_k \right\| = \sum_{k=n}^{n+m} |c_k| \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty$$

Hence $(y_n)_{n=1}^{\infty}$ where $y_n = \sum_{k=1}^n c_k x_k$ is a Cauchy sequence in A .

Thus we may define an operator $T: l_1 \rightarrow A$ by

$$T((c_n)) = \sum_{n=1}^{\infty} c_n x_n.$$

This is a linear operator and

$$\|T((c_n))\| = \|(c_n)\|$$

and the proof is complete.

DEFINITION. Let e be a linear projection in A . We shall say that e is an M-projection in A if

$$\|x\| = \max\{\|e(x)\|, \|x-e(x)\|\} \quad \text{all } x \in A.$$

A subspace J of A is said to be an M-summand if J is the

range of an M -projection.

Let e be a bounded linear projection in A and let e^* be the adjoint projection in A^* . Then Alfsen and Effros proved in [4] Proposition 5.15. that:

- (i) e is an M -projection if and only if e^* is an L -projection.
- (ii) e is an L -projection if and only if e^* is an M -projection.

This has the following consequences (See [4] Corollaries 5.16. to 5.20.)

- (a) M summands are M -ideals.
- (b) An M -summand is the range of only one M -projection.
- (c) The M -projections commute.
- (d) If A is reflexive, then every M -ideal is an M -summand.
- (e) If J is an M -ideal in A , then J is an M -summand if and only if there exists an M -ideal H with $J \cap H = (0)$ and $J + H = A$.

LEMMA 5.5. Let J and H be two M -summands in A . Then $J \cap H$ is an M -summand in H and

$$H = (J \cap H) \oplus (J' \cap H)$$

where J' is the M -ideal such that $J \cap J' = (0)$ and $J + J' = A$.

LEMMA 5.6. Let J be an M -summand in A and let H be an M -summand in J . Then H is an M -summand in A .

The proofs of Lemma 5.5. and Lemma 5.6. are easy and are omitted. (See the proof of Lemma 5.2.)

LEMMA 5.7. Suppose A contains infinitely many different M -summands. Then there exists a sequence $(J_n)_{n=1}^{\infty}$ of M -summands

in A such that $J_n \neq J_{n+1}$ all n and

$$A \supseteq J_1 \supseteq J_2 \supseteq \dots \supseteq J_n \supseteq J_{n+1} \supseteq \dots$$

Proof: The argument is similar to the argument used in the proof of Lemma 5.3.

THEOREM 5.8. Suppose A contains infinitely many different M -summands. Then there exists a linear operator $T: c_0 \rightarrow A$ such that

$$\|Tx\| = \|x\| \quad \text{all } x \in c_0.$$

Proof: The proof is similar to the proof of Theorem 5.4.

COROLLARY 5.9. Suppose A is reflexive. Then A contains only finitely many M -ideals.

Proof: Since M -ideals in reflexive Banach spaces are M -summands and no reflexive Banach space contains c_0 , the result follows from Theorem 5.8.

COROLLARY 5.10. Suppose A is reflexive. Then A contains only finitely many L -ideals.

Proof: From Corollary 5.9. A^* contains only finitely many M -ideals, so the result follows.

Let $p \in \partial_e A_1^*$. By Corollary 3.23. there exists a largest M -ideal, denoted J_p , in $\ker p = \{x \in A : p(x) = 0\}$. M -ideals of the form J_p for some $p \in \partial_e A_1^*$ will be said to be primitive.

The set of primitive ideals in A will be denoted $\text{prim } A$. Following Alfsen and Effros [4] we shall call a subset S of $\text{prim } A$ a hull if there exists an M -ideal J such that

$$S = h(J) = \{J_p \in \text{prim } A : J \subseteq J_p\}$$

If $S \subseteq \text{prim } A$ we define the kernel of S , $k(S)$, to be the largest M -ideal contained in $\cap \{J_p : J_p \in S\}$. Then Alfsen and Effros shows that the hulls form the closed sets for a topology on $\text{prim } A$, called the structure topology.

If $S \subseteq \text{prim } A$, then $\bar{S} = hk(S)$.

We can give the $\text{prim } A$ for some Banach spaces A . We assume A is real and $\dim A > 2$.

Space A	$\text{prim } A$
c_0	\mathbb{N}
c	$\mathbb{N} \cup \{\infty\}$
$l_p \quad (1 \leq p < \infty)$	(0)
l_∞	$\beta \mathbb{N}$
$C(X) \quad (X \text{ compact Hausdorff})$	X
$L^1(\mu)$	(0)

The result for l_1 and $L_1(\mu)$ follows from Corollary 3.16., Proposition 5.15. in [4] and [16].

If $A = l_q$, $1 < q < \infty$, and $x \in A$ with $\|x\| = 1$, then $x \in \partial_e A_1$. Hence if e is an L -projection in l_q , then $e(x) = 0$ or $e(x) = x$ by Corollary 2.10. This gives that $e = 0$ or $e = I$. But then l_p , $1 < p < \infty$, cannot contain any non-trivial M -ideal.

We know that c is isometric to $C(\mathbb{N} \cup \{\infty\})$, where $\mathbb{N} \cup \{\infty\}$ is the one-point compactification of the natural numbers \mathbb{N} , and

l_∞ is isometric to $C(\beta\mathbb{N})$, where $\beta\mathbb{N}$ is the Stone-Čech compactification of \mathbb{N} . c_0 is an M-ideal of c . (See [4] Proposition 9.18) It follows from [4] (See Proposition 9.18. and the remarks preceeding Proposition 5.15.) that if $A = C(X)$, then $\text{prim} A$ is homeomorphic to X .

6. A selection theorem.

Let E and F be Hausdorff locally convex vector spaces and let $K \subseteq E$ be a convex non-empty subset. Let F^c be the family of all compact convex non-empty subsets of F . A map $\varphi: K \rightarrow F^c$ is said to be convex if

$$\lambda\varphi(x) + (1-\lambda)\varphi(y) \subseteq \varphi(\lambda x + (1-\lambda)y) \quad \text{all } x, y \in K, \text{ all } \lambda \in [0, 1].$$

and φ is said to be affine if

$$\lambda\varphi(x) + (1-\lambda)\varphi(y) = \varphi(\lambda x + (1-\lambda)y) \quad \text{all } x, y \in K, \text{ all } \lambda \in [0, 1].$$

LEMMA 6.1. Let I be a directed set and let $(\varphi_\alpha)_{\alpha \in I}$ be a family of affine maps from K into F^c such that if $\alpha \leq \beta$ then $\varphi_\alpha(x) \supseteq \varphi_\beta(x)$ all $x \in K$. Define $\varphi: K \rightarrow F^c$ by $\varphi(x) = \bigcap_{\alpha \in I} \varphi_\alpha(x)$. Then φ is an affine map.

Proof: Clearly each $\varphi_\alpha(x)$ is compact convex and non-empty. Let $x, y \in K$ and $\lambda \in [0, 1]$.

Then for $\beta \in I$:

$$\lambda\varphi(x) + (1-\lambda)\varphi(y) \subseteq \lambda\varphi_\beta(x) + (1-\lambda)\varphi_\beta(y) = \varphi_\beta(\lambda x + (1-\lambda)y)$$

Hence

$$\lambda\varphi(x) + (1-\lambda)\varphi(y) \subseteq \varphi(\lambda x + (1-\lambda)y).$$

On the other hand, for $\alpha \in I$ we have

$$\varphi(\lambda x + (1-\lambda)y) \subseteq \varphi_{\alpha}(\lambda x + (1-\lambda)y) = \lambda \varphi_{\alpha}(x) + (1-\lambda) \varphi_{\alpha}(y)$$

The compactness of all sets in F^C now implies

$$\varphi(\lambda x + (1-\lambda)y) \subseteq \lambda \varphi(x) + (1-\lambda) \varphi(y)$$

and the proof is complete.

LEMMA 6.2. Assume K has the Riesz decomposition property and assume $\varphi: K \rightarrow F^C$ is a convex map. Then there exists an affine map $\psi: K \rightarrow F^C$ such that $\psi(x) \subseteq \varphi(x)$ all $x \in K$.

Proof: Define for each $x \in K$, $\psi(x) \subseteq F$ by

$$(*) \quad \psi(x) = \cap \left\{ \sum_{i=1}^n \lambda_i \varphi(x_i) : x_i \in K, \lambda_i \in [0,1], \sum_{i=1}^n \lambda_i = 1, x = \sum_{i=1}^n \lambda_i x_i \right\}$$

clearly $\psi(x)$ is compact and convex since each set of the form

$$\sum_{i=1}^n \lambda_i \varphi(x_i)$$

is convex and compact. We also have that if

$$x = \sum_{i=1}^n \lambda_i x_i, \quad x_i \in K, \quad \lambda_i \in [0,1], \quad \sum_{i=1}^n \lambda_i = 1$$

then

$$\sum_{i=1}^n \lambda_i \varphi(x_i) \subseteq \varphi\left(\sum_{i=1}^n \lambda_i x_i\right) = \varphi(x).$$

Hence $\psi(x) \subseteq \varphi(x)$.

The rest of the proof will be shown in three steps.

Step 1. $\psi(x) \neq \emptyset$

$$\text{Suppose } x = \sum_{i=1}^n \lambda_i x_i = \sum_{j=1}^m \alpha_j y_j$$

are two convex combinations in K . Since K has the Riesz decomposition property, there exists $\theta_{ij} \geq 0$ and $z_{ij} \in K$ such that $\sum_{i,j} \theta_{ij} = 1$ and

$$\lambda_i x_i = \sum_{j=1}^m \theta_{ij} z_{ij}, \quad \alpha_j y_j = \sum_{i=1}^n \theta_{ij} z_{ij}$$

Then we have $x = \sum_{i,j} \theta_{ij} z_{ij}$ and

$$\sum_{i,j} \theta_{ij} \varphi(z_{ij}) \subseteq \left[\sum_{i=1}^n \lambda_i \varphi(x_i) \right] \cap \left[\sum_{j=1}^m \alpha_j \varphi(y_j) \right]$$

Hence the sets which we take intersection over in (*), is directed by inclusion, so it follows that $\psi(x) \neq \emptyset$.

Step 2. For $x, y \in K$ and $\lambda \in [0, 1]$ we have

$$\lambda \psi(x) + (1-\lambda) \psi(y) \supseteq \psi(\lambda x + (1-\lambda)y)$$

Suppose we have convex combinations in K :

$$x = \sum_{i=1}^n \lambda_i x_i \quad \text{and} \quad y = \sum_{j=1}^m \alpha_j y_j$$

Then

$$\lambda x + (1-\lambda)y = \sum_{i=1}^n \lambda \lambda_i x_i + \sum_{j=1}^m (1-\lambda) \alpha_j y_j$$

so

$$\begin{aligned} \psi(\lambda x + (1-\lambda)y) &\subseteq \sum_{i=1}^n \lambda \lambda_i \varphi(x_i) + \sum_{j=1}^m (1-\lambda) \alpha_j \varphi(y_j) \\ &= \lambda \left[\sum_{i=1}^n \lambda_i \varphi(x_i) \right] + (1-\lambda) \left[\sum_{j=1}^m \alpha_j \varphi(y_j) \right] \end{aligned}$$

Since the sets which we take intersection over in (*), is directed by inclusion (See Step 1.) a simple compactness argument gives

$$\psi(\lambda x + (1-\lambda)y) \subseteq \lambda \psi(x) + (1-\lambda) \psi(y).$$

Step 3. For $x, y \in K$ and $\lambda \in [0, 1]$ we have

$$\lambda \psi(x) + (1-\lambda) \psi(y) \subseteq \psi(\lambda x + (1-\lambda)y)$$

Let $\lambda x + (1-\lambda)y = \sum_{j=1}^n \lambda_j x_j$ be a convex combination in K .

The Riesz decomposition property implies that there exists $\theta_{ij} \geq 0$ and $z_{ij} \in K$ such that $\sum_{i,j} \theta_{ij} = 1$ and

$$\lambda x = \sum_{j=1}^n \theta_{1j} z_{1j} , \quad (1-\lambda)y = \sum_{j=1}^n \theta_{2j} z_{2j} , \quad \lambda_j x_j = \sum_{i=1}^2 \theta_{ij} z_{ij} .$$

Then we have

$$\begin{aligned} \lambda \psi(x) + (1-\lambda)\psi(y) &\subseteq \sum_j \theta_{1j} \varphi(z_{1j}) + \sum_j \theta_{2j} \varphi(z_{2j}) \\ &= \sum_{i,j} \theta_{ij} \varphi(z_{ij}) \subseteq \sum_{j=1}^n \lambda_j \varphi\left(\sum_{i=1}^2 \theta_{ij} \lambda_j^{-1} z_{ij}\right) \\ &= \sum_{j=1}^n \lambda_j \varphi(x_j) \end{aligned}$$

Hence

$$\lambda \psi(x) + (1-\lambda)\psi(y) \subseteq \psi(\lambda x + (1-\lambda)y) ,$$

and the proof is complete.

THEOREM 6.3. Suppose K has the Riesz decomposition property and let $\varphi: K \rightarrow F^C$ be a convex map. Then φ has an affine selection, i.e. there exists an affine map $\psi: K \rightarrow F$ such that $\psi(x) \in \varphi(x)$ all $x \in K$.

Proof: By Lemma 6.2. there exists an affine map $\eta: K \rightarrow F^C$ such that $\eta(x) \subseteq \varphi(x)$ all $x \in K$. Let B be the set of all affine maps $\eta: K \rightarrow F^C$ such that $\eta(x) \subseteq \varphi(x)$ all $x \in K$. We partially order B by $\eta_1 \leq \eta_2$ if and only if $\eta_1(x) \supseteq \eta_2(x)$ all $x \in K$. Zorn's lemma together with Lemma 6.1. gives that B contains maximal elements. Let η be a maximal element in B . Then clearly $\eta(x) \subseteq \varphi(x)$ all $x \in K$. Suppose there exists $z \in K$ such that $\eta(z)$ consist of more than one element. We have

$$\text{face}(z) = \{y \in K: z = \lambda y + (1-\lambda)u \text{ for some } u \in K \text{ and some } \lambda \in \langle 0, 1 \rangle\} .$$

Then $\text{face}(z)$ is the smallest face of K containing z . (See [1]). Since $\eta(z)$ is compact and convex there exists a

$e(z) \in \partial_e \eta(z)$. If $z = \lambda y + (1-\lambda)u$ where $u \in K$ and $\lambda \in \langle 0,1 \rangle$, then there exists a unique $e(y) \in \eta(y)$ such that

$$e(z) = \lambda e(y) + (1-\lambda)e(u) .$$

Hence we can define $\Theta : K \rightarrow \mathbb{F}^C$ by

$$\Theta(x) = \begin{cases} e(y) & \text{if } y \in \text{face}(z) \\ \eta(x) & \text{if } x \notin \text{face}(z) \end{cases}$$

Then clearly Θ is convex. Using Lemma 6.2. on Θ now gives a contradiction to the maximality of η . This shows that $\eta(x)$ consists of one point for all $x \in K$, and the proof is complete.

COROLLARY 6.4. Let J be a closed subspace of a real Banach space A . If J is a Lindenstrauss space, then there exists a projection Q in A^* such that $(I-Q)A^* = J^\circ$ and $\|Q\| \leq 1$.

Proof: Since J is a Lindenstrauss space we have that J^* is isometric to a $L^1(\mu)$ space. Hence there exists a face F of J_1^* such that $J_1^* = \text{co}(FU-F)$ and F has the Riesz decomposition property. Define $\varphi : F \rightarrow (A^*)^C$ by

$$\varphi(f) = \{g \in A^* : \|g\| \leq \|f\| \text{ and } g|_J = f\} .$$

We may assume $\|f\| = 1$ all $f \in F$. Then

$$\varphi(f) = \{g \in A_1^* : g|_J = f\}$$

so $\varphi(f)$ is w^* -compact convex and non-empty for all $f \in F$.

It is clear that φ is convex. By Theorem 6.3. there exists an affine map $\psi : F \rightarrow A_1^*$ such that $\psi(f) \in \varphi(f)$ all $f \in F$.

Now ψ has a unique extension to a linear operator $T : J^* \rightarrow A^*$ and $T(J_1^*) \subseteq A_1^*$. Let $R : A^* \rightarrow J^*$ be the restriction map and define $Q : A^* \rightarrow A^*$ by

$$Q = T \circ R .$$

This Q has the desired properties, and the proof is complete.

COROLLARY 6.5. Let J be a closed subspace of a complex Banach space A . If J is a Lindenstrauss space, then there exists a projection Q in A^* such that $\ker Q = J^\circ$ and $\|Q\| \leq 1$.

Proof: We have that J^* is isometric to a space $L_1(\mu)$ for some measure μ . Let

$$F = \{f \in L_1(\mu) : f \geq 0 \text{ and } \|f\| = 1\}.$$

Then F is a maximal proper face of the unit ball of $L_1(\mu)$ and F has the Riesz decomposition property. As in Corollary 6.4. there exists an affine map $\psi : F \rightarrow A^*$ such that $\|\psi(f)\| = \|f\|$ all $f \in F$. For $f \in L_1(\mu)$ write

$$f = a_1 f_1 - a_2 f_2 + i a_3 f_3 - i a_4 f_4$$

where $a_i \geq 0$ and $f_i \in F$, $i = 1, \dots, 4$.

Define $\Phi : L_1(\mu) \rightarrow A^*$ by

$$\Phi(f) = a_1 \psi(f_1) - a_2 \psi(f_2) + i a_3 \psi(f_3) - i a_4 \psi(f_4)$$

It is easy to see that Φ is well-defined and linear. Let φ be the restriction map from A^* to J^* composed with the isometry from J^* to $L_1(\mu)$. Then we have

$$\varphi \circ \Phi(f) = f \quad \text{all } f \in L_1(\mu).$$

Define $Q : A^* \rightarrow A^*$ by

$$Q = \Phi \circ \varphi.$$

Obviously Q is a linear projection.

Since $\|\varphi\| = 1$ and $\varphi \circ \Phi = \text{id}$ on $L_1(\mu)$ we get that $\|f\| \leq \|\Phi(f)\|$ all $f \in L_1(\mu)$. This implies that $g \in \ker Q$ if and only if $\varphi(g) = 0$ i.e. $\ker Q = J^\circ$.

It only remains to prove that $\|Q\| \leq 1$. It is enough to show

that $\|\Phi(f)\| \leq \|f\|$ for $f \in L_1(\mu)$, so Φ is an isometry. We may assume that μ is defined on a set X . Let A_1, \dots, A_n be disjoint measurable sets in X with finite measure and let $c_1, \dots, c_n \in \mathbb{C}$. Define $f = \sum_{i=1}^n c_i \chi_{A_i} \in L_1(\mu)$. Since such elements are dense in $L_1(\mu)$ it is enough to show $\|\Phi(f)\| \leq \|f\|$ for this f . Now

$$\begin{aligned} \|\Phi(f)\| &= \left\| \sum_{i=1}^n c_i \Phi(\chi_{A_i}) \right\| \\ &\leq \sum_{i=1}^n |c_i| \|\Phi(\chi_{A_i})\| = \sum_{i=1}^n |c_i| \|\chi_{A_i}\| \\ &= \sum_{i=1}^n |c_i| \mu(A_i) = \|f\|. \end{aligned}$$

The proof is complete.

COROLLARY 6.6. Let K be a convex set with the Riesz decomposition property and let G be a compact face of K . Then there exists an affine map $\psi: K \rightarrow G$ such that $\psi(x) = x$ all $x \in G$.

Proof: Use Theorem 6.3. on the map φ defined by

$$\varphi(x) = \begin{cases} \{x\} & \text{if } x \in G \\ G & \text{if } x \in K \setminus G. \end{cases}$$

REMARK: Continuous maps of the type in Corollary 6.6. has been constructed by various authors. (See [24] and [7].)

REMARK: Theorem 6.3. can also be used to show that a compact convex set K is a Choquet simplex if and only if the barycenter map has an affine selection. (See [1] Theorem II.3.6, [28] Theorem 6, [23] Theorem 3.4., [15] and [27].)

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